[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1381 C

Unique Paper Code : 32351301

Name of the Paper : BMATH 305 – Theory of Real Functions

Name of the Course : CBCS (LOCF) B.Sc. (H) Mathematics

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. Attempt any **two** parts from each question.

3. **All** questions are compulsory.

1. (a) Let $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A and $f: A \to \mathbb{R}$, then define limit of function f at c.

   Use $\varepsilon - \delta$ definition to show that $\lim\limits_{x \to 1} \frac{x}{x+1} = \frac{1}{2}$.

   (6)

P.T.O.

(b) Let $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A. Then show that $\lim_{x \to c} f(x) = L$ if and only if for every sequence $\langle x_n \rangle$ in A that converges to c such that $x_n \neq c$, $\forall n \in \mathbb{R}$, the sequence $\langle f(x_n) \rangle$ converges to L. (6)

(c) Show that $\lim_{x \to 0} \sin\left(\frac{1}{x^2}\right)$ does not exist in $\mathbb{R}$ but $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x^2}\right) = 0$. (6)

2. (a) Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $g: A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A. Show that if f is bounded on a neighborhood of c and $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c} (fg)(x) = 0$. (6)

(b) Let $f(x) = e^{1/x}$ for $x \neq 0$, then find $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0^+} f(x)$. (6)

(c) Let $f: \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 2x & \text{if } x \text{ is rational} \\ x + 3 & \text{if } x \text{ is irrational} \end{cases}.$$

Find all the points at which f is continuous.

(6)

3. (a) Let $A \subseteq \mathbb{R}$ and let f and g be real valued functions on A. Show that if f and g are continuous on A then their product fg is continuous on A. Also, give examples of two functions f and g such that both are discontinuous at a point $c \in A$ but their product is continuous at c. (7½)

(b) State and prove Boundedness Theorem for continuous functions on a closed and bounded interval. (7½)

(c) State Maximum-Minimum Theorem. Let $I = [a,b]$ and $f : I \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for each x in I. Prove that there exists a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all x in I. (7½)

4. (a) Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ such that $f(x) \geq 0$ for all $x \in A$. Show that if f is continuous at $c \in A$, then $\sqrt{f}$ is continuous at c. (6)

(b) Show that every uniformly continuous function on $A \subseteq R$ is continuous on A. Is the converse true? Justify your answer. (6)

(c) Show that the function $f(x) = \frac{1}{x^2}$, $x \neq 0$ is uniformly continuous on $[a, \infty)$, for $a > 0$ but not uniformly continuous on $(0, \infty)$. (6)

5\. (a) Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions that are differentiable at c. Prove that if $g(c) \neq 0$, the function f/g is differentiable at n, and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$ (6)

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x| + |x + 1|$, $x \in \mathbb{R}$. Is f differentiable everywhere in $\mathbb{R}$? Find the derivative of *f* at the points where it is differentiable. (6)

(c) State Mean Value Theorem. If $f: [a, b] \to \mathbb{R}$ satisfies the conditions of Mean Value Theorem and $f'(x) = 0$ for all $x \in (a,b)$. Then prove that f is constant on [a, b]. (6)

6\. (a) Let I be an open interval and let $f : I \to \mathbb{R}$ have a second derivative on I. Then show that f is a convex function on I if and only if $f''(x) \geq 0$ for all $x \in I$. (6)

(b) Find the points of relative extrema of the functions $f(x) = |x^2 - 1|$, for $-4 \leq x \leq 4$. (6)

(c) Use Taylor's Theorem with n = 2 to approximate $\sqrt[3]{1 + x}$,, $x > -1$. (6)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1409 **C**

Unique Paper Code : 32351302

Name of the Paper : BMATH306 – Group Theory-I

Name of the Course : **B.Sc. (Hons) Mathematics**

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** questions are compulsory.

3. Attempt any **two** parts from each question from **Q2** to **Q6**.

4. In the question paper, given notations have their usual meaning unless until stated otherwise.

P.T.O.

1. Give short answers to the following questions. Attempt any **six**.

(i) What is the total no of rotations and total no of reflections in the dihedral group D_3? Describe them (rotations and reflections) in pictures or words. What can you say about the total no of rotations and total no of reflections in the dihedral group D_n?

(ii) Give one non- trivial, proper subgroup of GL(2, R). Is GL(2, R) a group under addition of matrices? Answer in few lines.

(iii) Let G be a group with the property that for any a, b, c in G,

ab = ca implies b = c. Prove that G is Abelian.

(iv) Give an example of a cyclic group of order 5. Show that a group of order 5 is cyclic.

(v) Prove that a cyclic group is Abelian. Is the converse true?

(vi) Find all subgroups of Z_{15}.

(vii) Prove that 1 and -1 are the only two generators of (Z,+). Give short answer in few lines.

(viii) "Z_n. $n \in N$, is always cyclic whereas U(n), $n \in N$; $n \geq 2$ may or may not be cyclic". Prove or disprove the statement in a few lines. (6×2=12)

2. (a) Let G = $\{a + b\sqrt{2}$ | a and b are rational nos not both zero$\}$

Prove that G is a group under ordinary multiplication. Is it Abelian or Non-Abelian? Justify your answer.

(b) Prove that a group of composite order has a non-trivial, proper subgroup.

(c) Prove that order of a cyclic group is equal to the order of its generator. (2×6.5=13)

3. (a) Prove that every permutation of a finite set can be written as a cycle or as a product of distinct cycles. (6)

(b) (i) In S_4, write a cyclic subgroup of order 4 and a non-cyclic subgroup of order 4.

(ii) Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 8 & 6 & 7 & 5 & 1 & 3 \end{bmatrix}$ and

$$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 4 & 5 & 1 & 8 & 3 & 2 & 6 \end{bmatrix}$$

Write α, β and αβ as product of 2-cycles. (3+3=6)

(c) (i) Let $|a| = 24$. How many left cosets of $H = \langle a^4 \rangle$ in $G = \langle a \rangle$ are there? Write each of them.

(ii) State Fermat's Little theorem. Also compute 5^{25} mod 7 and 11^{17} mod 7. (3+3=6)

4. (a) (i) Let H and K be two subgroups of a finite group. Prove that

HK ≤ G if G is Abelian.

(ii) Give an example of a group G and its two subgroups H and K (H≠K) such that HK is not a subgroup of G. (3+3.5=6.5)

(b) (i) Let G be a group and let Z (G) be the centre of G. If G/Z (G) is cyclic, prove that G is Abelian.

(ii) Let $|G| = pq$, p and q are primes. Prove that $|Z(G)| = 1$ or pq. (4+2.5=6.5)

(c) (i) Prove that a subgroup of index 2 is normal.

(ii) Let $G = U(32)$, $H = U_8(32)$. Write all the elements of the factor group G/H. Also find order of 3H in G/H. (3+3.5=6.5)

5. (a) Show that the mapping from $\mathbb{R}$ under addition to $GL(2, \mathbb{R})$ that takes x to $\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$ is a group homomorphism. Also, find the kernel of the homomorphism.

(b) Let ϕ be a homomorphism from a group G to a group $\bar{G}$. Show that if $\bar{K}$ is a subgroup of $\bar{G}$, then $\phi^{-1}(\bar{K}) = [k \in G: \phi(k) \in \bar{K}\}$ is a subgroup of G.

(c) If H and K are two normal subgroups of a group G such that $H \subseteq X$, then prove that

$$G/K \approx \frac{G/H}{K/H}.$$

(2×6=12)

6. (a) Show that the mapping ϕ from $\mathbb{C}^*$ to $\mathbb{C}^*$ given by $\phi(z) = z^4$ is a homomorphism. Also find the set of all the elements that are mapped to 2.

(b) Prove that every group is isomorphic to a group of permutations.

(c) Let G be the group of non-zero complex numbers under multiplication and N be the set of complex numbers of absolute value 1.

Show that G/N is isomorphic to the group of all the positive real numbers under multiplication.

(2×6.5=13)

[This question paper contains 4 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1427 C

Unique Paper Code : 32351303

Name of the Paper : BMATH 307 – Multivariate Calculus

Name of the Course : B.Sc. (H) Mathematics

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** sections are compulsory

3. Attempt any **Five** questions from each section. **All** questions carry equal marks

SECTION I

1. Let $f(x,y)=\dfrac{xy\left(x^2-y^2\right)x}{x^2+y^2}$ if $(x, y) \neq (0,0)$

$= 0$ otherwise

Show that $f(0, y) = -y$ and $f(x, 0) = x$ for all x and y.

2. Use incremental approximation to estimate the function $f(x, y) = \sin(xy)$ at the point

$$\left(\sqrt{\frac{\pi}{2}}+.01, \sqrt{\frac{\pi}{2}}-.01\right)$$

3. If $z = xy + f(x^2 + y^2)$, show that $y\,\partial z/\partial x - x\,\partial z/\partial y = y^2 - x^2$.

4. Assume that maximum directional derivative of f at $P_0(1,2)$ is equal to 50 and is attained in the direction towards $Q(3, -4)$. Find ∇f at $P_0(1, 2)$.

5. Find the absolute extrema of $f(x, y) = 2x^2 - y^2$ on the disk $x^2 + y^2 \leq 1$.

6. Use Lagrange multiplier to find the distance from (0, 0, 0) to plane $Ax + By + Cz = D$ where at least one of A, B, C is nonzero.

SECTION II

1. Compute the integral $\int_0^1 \int_x^{2x} e^{y-x}\,dy\,dx$ with the order of integration reversed.

2. Use Polar double integral to show that a sphere of radius α has volume $\frac{4}{3}\pi a^3$.

3. Compute the area of region D bounded above by line $y = x$, and below by circle $x^2 + y^2 - 2y = 0$.

4. Find the volume of the solid bounded above by paraboloid $z = 6 - x^2 - y^2$ and below by $z = 2x^2 + y^2$.

5. Evaluate $\iiint_D \frac{dx\,dy\,dz}{\sqrt{x^2 + y^2 + z^2}}$, where D is the solid sphere $x^2 + y^2 + z^2 \leq 3$.

6. Use a suitable change of variables to find the area of region R bounded by the hyperbolas $xy=1$ and $xy=4$ and the lines $y=x$ and $y=4x$.

SECTION III

1. Find the mass of a wire in the shape of curve C: $x = 3 \sin t$, $y = 3 \cos t$, $z = 2t$ for $0 \leq t \leq \pi$ and density at point (x, y, z) on the curve is $\delta(x, y, z) = x$.

2. Find the work done by force

$$\vec{F}(x,y,z) = (y^2 - z^2)\hat{i} + (2yz)\hat{j} - (x^2)\hat{k}$$

on an object moving along the curve C given by $x(t) = t$, $y(t) = t^2$, $z(t) = t^3$, $0 \leq t \leq 1$.

3. Use Green's theorem to find the work done by the force field

$$\vec{F}(x,y) = (3y - 4x)\hat{i} + (4x - y)\hat{j}$$

when an object moves once counterclockwise around the ellipse $4x^2 + y^2 = 4$.

4. Use Stoke's theorem to evaluate the surface integral

$$\iint_S (\text{curl } \vec{F}.N)\, dS$$

where $F = x\,i + y^2 j + z\,e^{xy}\, k$ and S is that part of surface $z = 1 - x^2 - 2y^2$ with $z \geq 0$.

5. Use divergence theorem to evaluate the integral $\iint_S \vec{F}.N\, dS$ where $\vec{F}(x,y,z) = (\cos yz)\hat{i} + e^{xz}\hat{j} + 3z^2\hat{k}$,

where S is hemisphere surface $z = \sqrt{4 - x^2 - y^2}$ together with the disk $x^2 + y^2 \leq 4$, in x-yplane.

6. Evaluate the line integral $\int_C \vec{F}.d\vec{R}$

Where $\vec{F}(x, y) = [(2x - x^2 y)\, e^{-xy} + \tan^{-1} y]\,\hat{i}$ + $\left[\frac{x}{y^2+1} - x^3 e^{-xy}\right]\hat{j}$ and C is the ellipse $9x^2 + 4y^2 = 36$.