Limits to Infinity:

Definition $\lim_{x\to\infty} f(x) = L$ means that for any number $\varepsilon > 0$, there exists a number N_1 such that

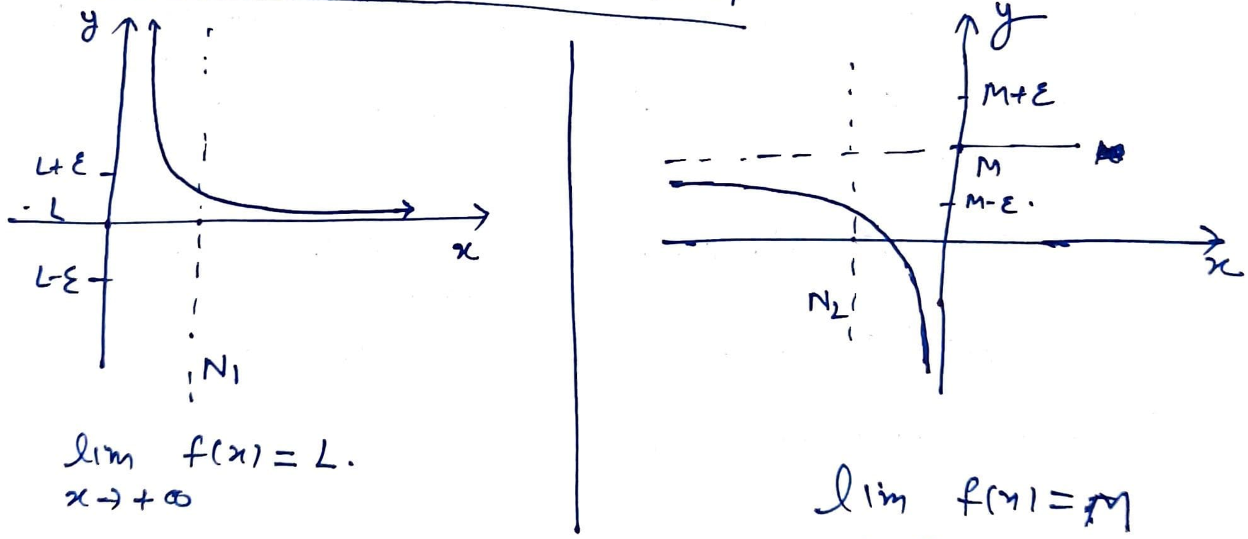
$$|f(x) - L| < \varepsilon \quad \text{whenever } x > N_1.$$

for x in the domain of f.

Similarly, $\lim_{x\to -\infty} f(x) = M$ means that for any $\varepsilon > 0$, there exists a number N_2 such that

$$|f(x) - M| < \varepsilon \quad \text{whenever } x < N_2.$$

Graphical Representation of limits to infinity.



$\lim_{x\to+\infty} f(x) = L.$ $\qquad$ $\lim_{x\to-\infty} f(x) = M$

Que. 1. Evaluate. $\lim_{x\to+\infty} \dfrac{3x^3 - 5x + 9}{5x^3 + 2x^2 - 7}$

Soln Dividing both the numerator and denominator of the given expressions by x^3, the highest power of x appearing in the fraction. we find

$$\frac{3x^3 - 5x + 9}{5x^3 + 2x^2 - 7} = \frac{x^3\left(3 - \frac{5}{x^2} + \frac{9}{x^3}\right)}{x^3\left(5 + \frac{2}{x} - \frac{7}{x^3}\right)}.$$

Thus

$$\lim_{x\to\infty} \frac{3x^3-5x+9}{5x^3+2x^2-7} = \lim_{x\to\infty} \frac{3-\frac{5}{x^2}+\frac{9}{x^3}}{5+\frac{2}{x}-\frac{7}{x^3}}$$

$$= \frac{\lim_{x\to\infty}\left(3-\frac{5}{x^2}+\frac{9}{x^3}\right)}{\lim_{x\to\infty}\left(5+\frac{2}{x}-\frac{7}{x^3}\right)}$$

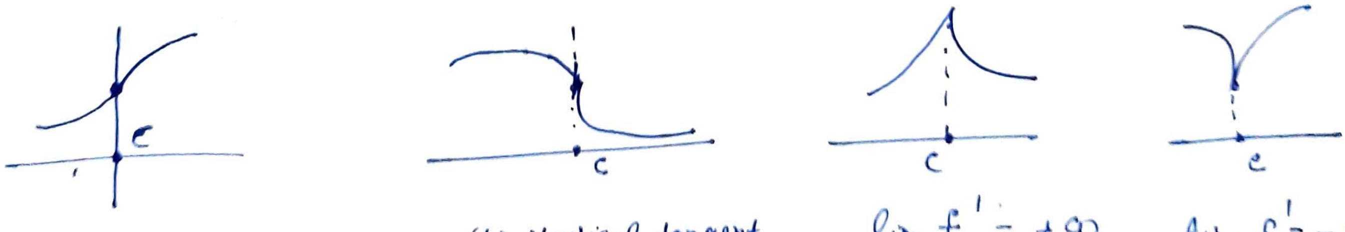
$$= \frac{3-0+0}{5+0-0} = \frac{3}{5}.$$

Vertical Tangents and Cusps.

" Suppose the function f is continious. at the point $P(c, f(c))$. The graph of f has vertical tangent at P if $\lim_{x\to c^-} f'(x)$ and $\lim_{x\to c^+} f'(x)$ are either both $+\infty$ or both $-\infty$.

" A Cusp P if $\lim_{x\to c^-} f'(x)$ and $\lim_{x\to c^+} f'(x)$ are both infinite with appositive signs (one $+\infty$ and other $-\infty$).

These possibilities are shown in figure



(a) Vertical tangent

$\lim_{x\to c^-} f'(x) = +\infty$

$\lim_{x\to c^+} f'(x) = +\infty$

(b) Vertical tangent

$\lim_{x\to c^-} f'(x) = -\infty$

$\lim_{x\to c^+} f'(x) = -\infty$

$\lim_{x\to c^-} f' = +\infty$

$\lim_{x\to c^+} f' = -\infty$

$\lim_{x\to c^-} f' = -\infty$

$\lim_{x\to c^+} f' = +\infty$

Infinite limits:

Def. $\lim_{x\to c} f(x) = +\infty$ if for any number $N>0$ (no matter how large) it is possible to find a number $\delta>0$ such that $f(x)>N$ whenever $0<|x-c|<\delta$.

Similarly $\lim_{x\to c} f(x) = -\infty$ if for any $N>0$, it is possible to find a number $\delta>0$ so that $f(x)<-N$ when $0<|x-c|<\delta$.

Ex Find $\lim_{x\to2^-} \frac{3x-5}{x-2}$ and $\lim_{x\to2^+} \frac{3x-5}{x-2}$

Sol^n

Notice that

$$\lim_{x\to2^+} \frac{1}{x-2} = +\infty \quad \text{and} \quad \lim_{x\to2^-} \frac{1}{x-2} = -\infty.$$

We also have $\lim_{x\to2} (3x-5) = 1$. and it follows that

$$\lim_{x\to2^+} \frac{3x-5}{x-2} = +\infty. \quad \text{and} \quad \lim_{x\to2^+} \frac{3x-5}{x-2} = -\infty.$$

Asymptotes:

Def. " The line $x=c$ is a vertical asympate of the graph of f if either of the one sided limits

$$\lim_{x\to c^-} f(x) \quad \text{or} \quad \lim_{x\to c^+} f(x)$$

is infinite.

" The line $y=L$ is horizontal asymptotes of the graph of f if

$$\lim_{x\to+\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

Que. Identify vertical tangents and cusps.

(a) $f(x) = x^{2/3}(2x+5$

(b) $g(x) = x^{1/3}(x+4)$

Solⁿ. (a) $f(x) = 2x^{5/3} + 5x^{2/3}$

$$f'(x) = \frac{10}{3}x^{2/3} + \frac{10}{3}x^{-1/3}$$

$$= \frac{10}{3}x^{-1/3}(x+1)$$

So $f'(x)$ becomes infinite only at $x = 0$.

Since

$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^-} \frac{10}{3}x^{-1/3}(x+1) = -\infty$$

and

$$\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} \frac{10}{3}x^{-1/3}(x+1) = +\infty.$$

it follows there is a cusp on the graph of f at $(0,0)$.

(b) $g'(x) = \frac{4}{3}x^{-2/3}(x+1)$

becomes infinite only when $x = 0$. We find that

$$\lim_{x \to 0^-} g'(x) = \lim_{x \to 0^-} \frac{4}{3}x^{-2/3}(x+1) = +\infty.$$

and $$\lim_{x \to 0^+} g'(x) = \lim_{x \to 0^+} \frac{4}{3}x^{-2/3}(x+1) = +\infty.$$

Thus vertical tangent occur at the origin..

References:-

1. Anton, Howard, Bivens Irl & Davis Stephen (2013), Calculus; John Wiley & Sons Singapore Pvt. Ltd.

2. Strauss M. J., Bradley, G. L. & Smith K. J. (2007), Calculus (3rd ed.), Dorling Kindersley (India) Pvt Ltd.

3. Thomas Jr. George B, Weier Maurice D. & Hass, Joel (2014), Thomas' Calculus (13th ed.), Pearson education Delhi