Subject :- Real Analysis
Course :- BSc. Phy. Science
Sem : IV.
Dept. : Mathematics.

Unit III

Riemann Integration.

We shall be dealing with closed finite intervals $[a, b]$ so that $(b-a) \in \mathbb{R}$ and $x \in [a, b]$ implies $a \leq x \leq b$. Moreover, all functions f will be assumed to be real-valued function defined and bounded on $[a, b]$.

Thus $f : [a, b] \longrightarrow \mathbb{R}$ and $|f(x)| \leq K$, where k is a positive real number.

Partition of a Closed Interval

Definition:- "A finitie ordered set of points $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is called a partition of $I=[a, b]$ if

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

The $(n+1)$ points $x_0, x_1, \ldots, x_n$ are called partition points of P.

The Interval $I_r = [x_{r-1}, x_r]$ is called the r-th subinterval of P.

These subintervals are clearly non-overlapping, except at the end points and they cover $[a, b]$, which justifies the term partition.

Clearly $$\bigcup_{r=1}^{n} I_r = \bigcup_{r=1}^{n} [x_{r-1}, x_r] = [a, b] = I$$

The length of the rth subinterval $I_r = [x_{r-1}, x_r]$ is denoted by δ_r. Thus $\delta_r = x_r - x_{r-1}$.

Note:1 By changing the partition points, the partition can be changed and hence, there can be infinite number of partitions of the interval $I = [a, b]$.

Note:2. We shall denote the set of all partitions of $[a, b]$ by $\mathcal{P}[a, b]$.

Norm of a partition:

Definition: "The maximum of the lengths of the subintervals of a partition P is called the **norm** or **mesh** of the partition P and is denoted by $\|P\|$.

Thus

$$\|P\| = \max\{\delta_r : r = 1, 2, 3, \dots n\}$$
$$= \max\{x_r - x_{r-1} : r = 1, 2, 3, \dots, n\}.$$

Refinement of a Partition

" If P and P' be two partitions of $[a, b]$ and $P \subset P'$, then the partition P' is called a refinement of partition P on $[a, b]$. We also say P' is finer than P.

Thus, if P' is finer than P, then every point of P is used in the construction of P' and P' has at least one additional point.

Remark: " If P_1 & P_2 are two partions of $[a, b]$, then $P_1 \subset P_1 \cup P_2$ and $P_2 \subset P_1 \cup P_2$. Therefore, $P_1 \cup P_2$ is called a common refinement of P_1 and P_2.

Note: 1. If $P_1, P_2 \in \mathcal{P}[a,b]$ and $P_1 \subset P_2$, then $\|P_2\| \leq \|P_1\|$.

Note: 2. If $P = \{x_0, x_1, x_2, \dots, x_n\}$ is a partition of $[a,b]$,

then
$$\sum_{r=1}^{n} \delta_r = \delta_1 + \delta_2 + \dots + \delta_n$$
$$= (x_1 - x_0) + (x_2 - x_1) + \dots + (x_n - x_{n-1})$$
$$= x_n - x_0$$
$$= b - a.$$

Upper and Lower Darbaux Sums.

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, b]$.

Since f is bounded on $[a, b]$, f is also bounded on each of the sub-intervals. Let M and m be the supremum and infimum of f on $[a, b]$ and let M_r and m_r be the suprimum and infimum of f in the r^{th} sub-interval

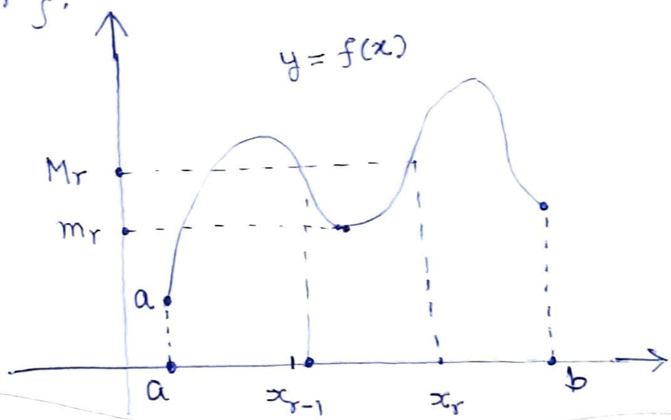
$$I_r = [x_{r-1}, x_r] \; ; \; r = 1, 2, 3, \cdots, n.$$

i.e.

$$M_r = \sup \{ f(x) \; : \; x \in I_r \}.$$

and

$$m_r = \inf \{ f(x) \; : \; x \in I_r \}.$$



" The upper Darboux sum $U(P, f)$ of the function f corresponding to the partition P is defined as.

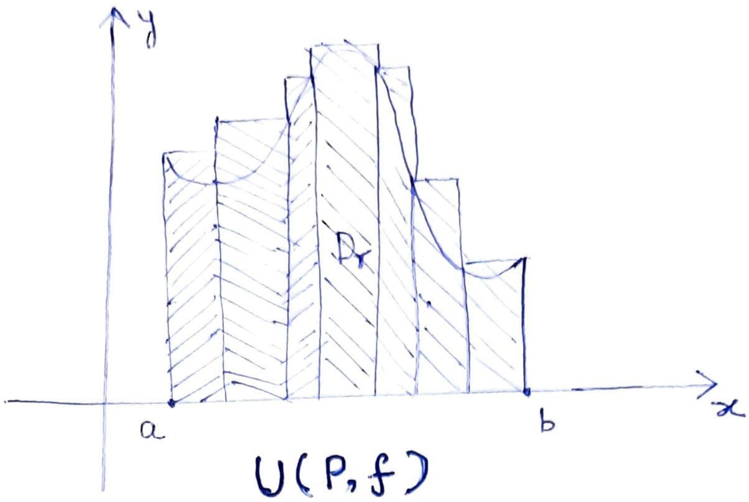
$$\boxed{U(P, f) = \sum_{r=1}^{n} M_r \cdot \delta_r = \sum_{r=1}^{n} M_r (x_r - x_{r-1})}$$

" The lower Darboux sum $L(P, f)$ of the function f corresponding to the partition P is defined as

$$\boxed{L(P, f) = \sum_{r=1}^{n} m_r \cdot \delta_r = \sum_{r=1}^{n} m_r \cdot (x_r - x_{r-1})}$$

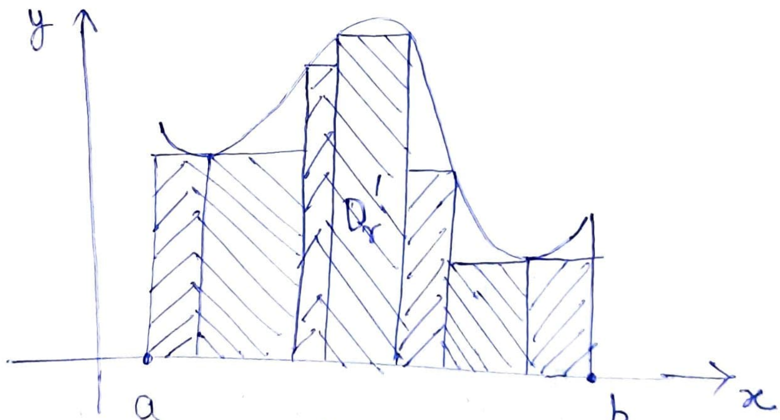
Note: Clearly, these sums depends upon the function f and the partition P and do exist for every bounded function.

Note 2: "If $f \geq 0$, then $U(P, f)$ is the are of the union of the rectangles D_r, D_r being the rectangle whose base is the subinterval $[x_{r-1}, x_r]$ and whose height M_r.



$U(P, f)$

Note: 3. If $f \geq 0$, then $L(P, f)$ is the are of the union of the rectangles D_r', where D_r' has the same base as D_r but has height m_r.



$L(P, f)$

Since $M_r \geq m_r$ and $x_r - x_{r-1} > 0$ for all r, we see that

$\boxed{U(P, f) \geq L(P, f)}$ for all $P \in \mathcal{P}[a, b]$.

Theorem: "If $f : [a,b] \longrightarrow \mathbb{R}$ is a bounded function and $P \in P[a,b]$, then

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

where m and M are the infimum and suprimum of f on $[a,b]$.

Proof: Let $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ be a partition of $[a,b]$.

Since f is bounded on $[a,b]$

$\Rightarrow f$ is bounded on each sub-interval $[x_{r-1}, x_r]$, $r = 1, 2, \dots, n$.

Let m_r and M_r be the infimum and suprimum of f on $[x_{r-1}, x_r]$.

Clearly,
$$m \le m_r \le M_r \le M.$$

$$\Rightarrow m \cdot \delta_r \le m_r \cdot \delta_r \le M_r \cdot \delta_r \le M \cdot \delta_r$$

where $\delta_r = x_r - x_{r-1}$, $r = 1, 2, \dots n$.

$$\Rightarrow \sum_{r=1}^{n} m \cdot \delta_r \le \sum_{r=1}^{n} m_r \cdot \delta_r \le \sum_{r=1}^{n} M_r \cdot \delta_r \le \sum_{r=1}^{n} M \cdot \delta_r$$

$$\Rightarrow m \cdot \sum_{r=1}^{n} \delta_r \le L(P,f) \le U(P,f) \le M \cdot \sum_{r=1}^{n} \delta_r$$

$$\Rightarrow m \cdot (b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

since $\sum_{r=1}^{m} \delta_r = b-a$

Hence proved

Note:- "The above theorem implies that $L(P,f)$ and $U(P,f)$ are bounded if f is bounded."

Theorem: If $f:[a,b] \to \mathbb{R}$ is a bounded function and $P, P' \in \mathcal{P}[a,b]$ such that $P \subset P'$ (i.e. P' is a refinement of P), then.

(i) $L(P,f) \le L(P',f)$.

(ii) $U(P,f) \ge U(P',f)$.

Proof: Let P' contain just one point ξ (say) more than

$$P = \{a = x_0, x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_n = b\}.$$

and $x_{r-1} < \xi < x_r$, then

$$P' = \{a = x_0, x_1, x_2, \ldots, x_{r-1}, \xi, x_r, \ldots, x_n = b\}.$$

Let m_r', m_r'' and m_r be the infimum of f in the intervals $[x_{r-1}, \xi]$, $[\xi, x_r]$ and $[x_{r-1}, x_r]$ respectively.

Let M_r', M_r'' and M_r be the supremum of f in the intervals. $[x_{r-1}, \xi]$, $[\xi, x_r]$ and $[x_{r-1}, x_r]$ respectively.

Then

$$m_r \le m_r', \qquad m_r \le m_r''$$

$$M_r \ge M_r', \qquad M_r \ge M_r''$$

(i) The contribution to $L(P,f)$ and $L(P',f)$ of each interval except $[x_{r-1}, x_r]$ is the same. Also the contribution of the sub-interval $[x_{r-1}, x_r]$ to $L(P,f)$ is $m_r(x_r - x_{r-1})$. and its contribution to $L(P',f)$ is

$$m_r'(\xi - x_{r-1}) + m_r''(x_r - \xi)$$

$$\therefore \quad L(P',f) - L(P,f) = [m_r' \cdot (\xi - x_{r-1}) + m_r''(x_r - \xi)] - m_r(x_r - x_{r-1})$$

$$= m_r' \cdot (\xi - x_{r-1}) + m_r''(x_r - \xi) - m_r \cdot [(x_r - \xi) + (\xi - x_{r-1})].$$

$$= (m_r' - m_r)(\xi - x_{r-1}) + (m_r'' - m_r)(x_r - \xi).$$

$$\ge 0$$

$[\because m_r' \ge m_r,\ m_r'' \ge m_r,\ x_{r-1} < \xi < x_r]$

$\Rightarrow L(P',f) \ge L(P,f)$

$\Rightarrow L(P,f) \le L(P',f)$

Hence proved

(ii) Try it.

Imp

Upper and Lower Riemann Integrals

" Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function. Then for every $P \in \mathcal{P}[a,b]$, we have

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

Where m and M are the infimum and suprimum of f on $[a,b]$.

Thus for every $P \in P[a,b]$, we have $L(P,f) \leq M(b-a)$

and $U(P,f) \geq m(b-a)$

$\Rightarrow$ The set $\{L(P,f)\}_{P \in P[a,b]}$ of lower sums is bounded above by $M(b-a)$ and therefore has the least upper bound.

The set $\{U(P,f)\}_{P \in P[a,b]}$ of ~~up~~ Upper sums is bounded below by $m(b-a)$ and ~~theref~~ therefore, has the greatest lower bound.

" Lower Riemann Integral of f on $[a,b]$ is as $\sup\{L(P,f)\}_{P \in P[a,b]}$ and is denoted by $\underline{\int_a^b} f(x)\,dx$.

i.e.

$$\boxed{\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) : P \in P[a,b]\}}$$

" Upper Riemann Integral of f on $[a,b]$ is defined as ~~$\inf\{U(P,f)\}_{P \in P[a,b]}$~~ $\inf\{U(P,f)\}_{P \in P[a,b]}$ and it is denoted by $\overline{\int_a^b} f(x)\,dx$.

Thus

$$\boxed{\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) : P \in P[a,b]\}}$$

where $P[a,b]$ = set of all partition of $[a,b]$.

Riemann Integral :

"Let $f : [a,b] \longrightarrow \mathbb{R}$ be bounded. We say that f is Riemann Integrable over $[a,b]$ if its lower and upper Riemann Integrals are equal i.e. if $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$.

The common value of these integrals is called the Riemann Integral of f on $[a,b]$ and is denoted by

$$\int_a^b f(x)\,dx.$$

Note: 1. Riemann Integrable (or simpley R-Integralble) on $[a,b]$.

Note-2. The family of all bounded functions which are R-Integrable on $[a,b]$ is denoted by $R[a,b]$.

Note 3. f is R-Integrable on $[a,b]$

$\Rightarrow$ (i) f is bounded on $[a,b]$

(ii) $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx = \int_a^b f(x)dx$.

Note 4. Previous figures (see pages No 4) suggest that no matter how we define the area A under the graph of f, we must have

$$L(P,f) \le A \le U(P,f).$$

Consequently we must have

$$\underline{\int_a^b} f(x)dx \le A \le \overline{\int_a^b} f(x)dx.$$

and so, if $f \in R[a,b]$, then A has to equal $\int_a^b f(x)\,dx$. In other words, the Integral $\int_a^b f(x)dx$ is a measure of the are under the curve of $f(x)$ from $x = a$ to $x = b$.

Example: Suppose f is a constant function.

$f(x) = c$ for all $x \in [a,b]$. Then show that $f \in R[a,b]$.

Sol^n Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of $[a,b]$.
Then for any sub-interval $[x_{r-1}, x_r]$, $r = 1, 2, \ldots, n$ we have

$$M_r = m_r = c.$$

where $M_r \to$ suprimum of f on $[x_{r-1}, x_r]$
$m_r \to$ infimum of f on $[x_{r-1}, x_r]$.

$$\therefore U(P,f) = \sum_{r=1}^{n} M_r.(x_r - x_{r-1}).$$
$$= \sum_{r=1}^{n} c\,(x_r - x_{r-1})$$
$$= c.\sum_{r=1}^{n} (x_r - x_{r-1})$$
$$= c\,(b-a)$$
$$= \text{constant}$$

and
$$L(P,f) = \sum_{r=1}^{n} m_r.(x_r - x_{r-1}) = c\sum_{r=1}^{n} (x_r - x_{r-1})$$
$$= c.(b-a)$$
$$= \text{constant}.$$

$$\therefore \underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) : P \in P[a,b]\}$$
$$= c(b-a) \quad \text{——— (1)}$$

Also
$$\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) : P \in P[a,b]\}$$
$$= c(b-a) \quad \text{——— (2)}$$

From (1) & (2)

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

Hence $f \in R[a,b]$ and $\int_a^b f(x)\,dx = c(b-a)$.

Ex If f is defined on $[0,1]$ by $f(x) = x \ \forall x \in [0,1]$, then $f \in R[0,1]$ and $\int_0^1 f(x)\,dx = \frac{1}{2}$.

Solⁿ.

Let $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{r-1}{n}, \frac{r}{n}, \ldots, \frac{n}{n} = 1\right\}$ be any partition of $[0,1]$. Then for any subinterval

$$I_r = \left[\frac{r-1}{n}, \frac{r}{n}\right], \quad r = 1, 2, 3, \ldots, n.$$

We have

$$M_r = \sup\{f(x) : x \in I_r\} = \frac{r}{n}$$

$$m_r = \inf\{f(x) : x \in I_r\} = \frac{r-1}{n}$$

and

$$\delta_r = \text{length of the interval } I_r = \frac{r}{n} - \frac{r-1}{n} = \frac{1}{n}.$$

$$\therefore \quad U(P,f) = \sum_{r=1}^{n} M_r \cdot \delta_r = \sum_{r=1}^{n} \frac{r}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \sum_{r=1}^{n} r = \frac{1}{n^2} \cdot \frac{n \cdot (n+1)}{2} = \frac{n+1}{2n}.$$

and

$$L(P,f) = \sum_{r=1}^{n} m_r \cdot \delta_r = \frac{1}{n^2} \sum_{r=1}^{n} (r-1) = \frac{1}{n^2} \cdot \frac{(n-1)\cdot(n)}{2} = \frac{n-1}{2n}$$

Therefore

$$\underline{\int_0^1} f(x)\,dx = \sup\{L(P,f) : P \in \mathcal{P}[0,1]\} = \lim_{n\to\infty}\left[\frac{n-1}{2n}\right] = \lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{n}\right) = \frac{1}{2}$$

and

$$\overline{\int_0^1} f(x)\,dx = \inf\{U(P,f) : P \in \mathcal{P}[0,1]\} = \lim_{n\to\infty}\left(\frac{n+1}{2n}\right) = \frac{1}{2}$$

Thus $\underline{\int_0^1} f = \overline{\int_0^1} f$. Hence $f \in R[0,1]$ and $\int_0^1 f(x) = \frac{1}{2}$. Ans

Ex If f is defined on $[0, a]$; $a > 0$ by

$$f(x) = x^2 \quad \forall\, x \in [0, a]$$

then $f \in R[0, a]$ and $\int_0^a f(x)\,dx = \frac{a^3}{3}$

Soln. Please try yourself.

Ex. Let $f(x) = \sin x$ for $x \in \left[0, \frac{\pi}{2}\right]$ and let $P = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \dots, \frac{n\pi}{2n}\right\}$ be the partition of $\left[0, \frac{\pi}{2}\right]$.

Compute $U(P,f)$, $L(P,f)$. Hence prove that $f \in R\left[0, \frac{\pi}{2}\right]$.

Soln. Here $P = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \dots, \frac{n\pi}{2n}\right\}$.

for any subinterval

$$I_r = \left[\frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}\right], \quad r = 1, 2, 3, \dots$$

$$M_r = \sin\left(\frac{r\pi}{2n}\right)$$

$$m_r = \sin\left(\frac{(r-1)\pi}{2n}\right) \qquad \left[\because f(x) = \sin x \text{ is increasing on } \left[0, \frac{\pi}{2}\right]\right]$$

$$\delta_r = \frac{r\pi}{2n} - \frac{(r-1)\pi}{2n} = \frac{\pi}{2n}.$$

$$U(P,f) = \sum_{r=1}^{n} M_r \cdot \delta_r = \frac{\pi}{2n} \sum_{r=1}^{n} \sin\left(\frac{r\pi}{2n}\right)$$

$$= \frac{\pi}{2n}\left[\sin\left(\frac{\pi}{2n}\right) + \sin\left(\frac{2\pi}{2n}\right) + \dots + \sin\left(\frac{n\pi}{2n}\right)\right]$$

$$= \frac{\pi}{2n} \cdot \frac{\sin\left(\frac{\pi}{2n} + \frac{n-1}{2} \cdot \frac{\pi}{2n}\right) \cdot \sin\left(\frac{n}{2} \cdot \frac{\pi}{2n}\right)}{\sin\left(\frac{1}{2} \cdot \frac{\pi}{2n}\right)}$$

$$\boxed{\because \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots \text{ to } n \text{ terms} = \frac{\sin\left(\alpha + \frac{(n-1)}{2}\cdot\beta\right) \cdot \sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}}$$

$$= \frac{\pi}{2n} \cdot \frac{\sin\frac{(n+1)\pi}{4\pi} \cdot \sin\frac{\pi}{4}}{\sin\left(\frac{\pi}{4n}\right)}$$

$$= \frac{\pi}{4n} \cdot \left[\cot\left(\frac{\pi}{4n}\right) + 1\right] \qquad [\text{after solving}].$$

Similarly

$$L(P,f) = \sum_{r=1}^{n} m_r \cdot \delta_r$$

$$= \frac{\pi}{4n}\left[\cot\frac{\pi}{4n} - 1\right]$$

Now

$$\underline{\int_{a}^{\pi/2}} f(x)\,dx = \sup\{L(P,f) : P \in \mathcal{P}[a,b]\}$$

$$= \lim_{n\to\infty} \frac{\pi}{4n}\cdot\left[\cot\frac{\pi}{4n} - 1\right]$$

$$= \lim_{n\to\infty}\left[\frac{\frac{\pi}{4n}}{\tan\frac{\pi}{4n}} - \frac{\pi}{4n}\right]$$

$$= 1.$$

and

$$\overline{\int_{0}^{\pi/2}} f(x)\,dx = \inf\{U(P,f) : P \in \mathcal{P}[a,b]\}$$

$$= \lim_{n\to\infty} \frac{\pi}{4n}\left[\cot\frac{\pi}{4n} + 1\right] = 1.$$

Since $\underline{\int_{0}^{\pi/2}} f(x)\,dx = \overline{\int_{0}^{\pi/2}} f(x)\,dx = 1.$

Hence $f \in R[0,\frac{\pi}{2}]$ and $\int_{0}^{\pi/2} f(x) = 1.$

Que Let $f : [a,b] \to \mathbb{R}$ defined by.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}\cap[a,b] \\ 0 & \text{if } x \notin \mathbb{Q}\cap[a,b]. \end{cases}$$ → Dirichlet function.

Let $P = \{x_0, x_1, \dots x_n\}$ be any partition of $[a,b]$.

We have

$m_r = 0$ for $r = 1,2,3,\dots$

$M_r = 1$

$$U(P,f) = \sum_{r=1}^{n} M_r\cdot(x_r - x_{r-1}) = 1\cdot(b-a) = b-a$$

$$L(P,f) = \sum_{r=1}^{n} m_r\cdot(x_r - x_{r-1}) = 0.$$

so $\overline{\int_{a}^{b}} f(x)\,dx = \inf\{U(P,f) : P \in \mathcal{P}[a,b]\} = b-a$

& $\underline{\int_{a}^{b}} f(x)\,dx = 0$

Thus $f \notin R[a,b]$. i.e. f is not R-Integrable.

References:

1. T. M. Apostol, Calculus, Volume 1, John Wiley and Sons, 2002.

2. R. G. Bartley and D. R. Sherbert, Introduction to real analysis, John Wiley and Sons, 2000.