Section 9.4 (Series of functions)

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22/03/2020

:- $\sum_{n=1}^{\infty} f_n$ = series of functions

$f_n : D \subseteq \mathbb{R} \to \mathbb{R}$

For each $x \in D$

$\sum f_n(x)$ is a series of real number.

- $\langle S_n \rangle$ is a series of partial sum where

$$S_n = x_1 + x_2 + x_3 + \cdots + x_n$$

&

$$l = \sum_{n=1}^{\infty} x_n$$

OR

$\langle S_n \rangle \to$ sequence of partial sums

where $S_n = f_1 + f_2 + \cdots + f_n$

and each S_n is a function s.t. $S_n : D \to \mathbb{R}$

For each $x \in D$, $f(x) = l(x) = \lim S_n(x) \quad \forall x \in D$

$$\sum f_n(x) = f(x)$$

i.e. $\sum f_n \to f$

Definition: 9.4.1 :- If $\langle f_n \rangle$ is a seq^n of funct^n defined on a subset D of $\mathbb{R}$ with values in $\mathbb{R}$, the seq^n of partial sums $\langle S_n \rangle$ of the infinite series $\sum f_n$ is defined for x in D by

$$S_1(x) = f_1(x)$$

$$S_2(x) = S_1(x) + f_2(x) = f_1(x) + f_2(x)$$

$$S_{n+1}(x) = S_n(x) + f_{n+1}(x) = f_1(x) + f_2(x) + \cdots + f_{n+1}(x)$$

In case, the seq^n $\langle S_n \rangle$ of functions converges on D to a function f, we say that the infinite series of function $\sum f_n$ converges to f on D.

* Further if the series $\sum |f_n(x)|$ converges for each x in D, we say that $\sum f_n$ is absolutely convergent on D.

* If the seq^n $\langle S_n \rangle$ of partial sums is uniformly cgt on D to f, we say that $\sum f_n$ is uniformly convergent on D, Or i.e. it converges to f uniformly on D.

(Statement only)

Theorem 9.4.2 :- If f_n is continuous on $D \subseteq \mathbb{R}$ to $\mathbb{R}$ for each $n \in \mathbb{N}$ and if $\sum f_n$ converges to f uniformly on D, then f is continuous on D.

Proof :- Thm 8.2.2 :- Let $\langle f_n \rangle$ be a seq^n of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that $\langle f_n \rangle$ converges uniformly on A to a function $f: A \to \mathbb{R}$. Then f is continuous on A.

(Statement only)

Theorem 9.4.3 :- Suppose that the real valued functions f_n, $n \in \mathbb{N}$, are Riemann integrable on the interval $J = [a,b]$. If the series $\sum f_n$ converges to f uniformly on J, then f is Riemann integrable and

$$\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$$

(Statement only)

Theorem 9.4.4 :- For each $n \in \mathbb{N}$, let f_n be a real-valued function on $J = [a,b]$ that has a derivative f_n' on J. Suppose that the series $\sum f_n$ converges for at least one point of J and that the series of derivatives $\sum f_n'$ converges uniformly on J.

Then there exists a real-valued function f on J such that $\sum f_n$ converges uniformly on J to f. In addition, f has a derivative on J and $f' = \sum f_n'$

— x —

* Cauchy's Criteria for Uniform Convergence of Series of function *

<u>Theorem 9.4.5</u> Let $\langle f_n \rangle$ be a seq^n of functions on $D \subseteq \mathbb{R}$ to $\mathbb{R}$. The series $\sum f_n$ is uniformly convergent on D iff for every $\epsilon > 0$ there exist an $M(\epsilon)$ such that if $m > n \geq M(\epsilon)$, then

$$|f_{n+1}(x) + \cdots + f_m(x)| < \epsilon \quad \forall \; x \in D.$$

<u>Proof</u> :–

Firstly, we will ~~state~~ state and prove cauchy criterion for uniform convergence, i.e. thm 8.1.10

Now we will prove the theorem.

let $\langle S_n \rangle$ be the seq^n of partial sums of the infinit series given by :–

$$S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x) \quad ; \forall x \in D$$

By Cauchy Criterion for seq^n of functions, we have $\langle S_n \rangle$ is uniformly convergent on D.

$\Longleftrightarrow$ for each $\epsilon > 0$, $\exists$ a natural no. $M(\epsilon)$ s.t.

$$|S_m(x) - S_n(x)| < \epsilon \quad \forall \; m > n \geq M(\epsilon)$$
$$\forall x \in D$$

$$\Leftrightarrow \left| \sum_{i=1}^{m} f_i(x) - \sum_{i=1}^{n} f_i(x) \right| < \epsilon \quad \forall \; m > n \geq M(\epsilon)$$
and $x \in D$.

$$\Leftrightarrow \quad |f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x)| < \epsilon \quad \forall\, m > n \geq M(\epsilon) \;\&\; \forall\, x \in D.$$

i.e. $\sum f_n$ is uniformly convergent on D iff for each $\epsilon > 0$, $\exists$ a natural no. $M(\epsilon)$ s.t.

$$|f_{n+1}(x) + \cdots + f_m(x)| < \epsilon \quad \forall\, m > n \geq M(\epsilon) \quad \forall\, x \in D$$

Hence proved

Theorem: 9.4.6 :– "Weierstrass M-test".

Let $\langle M_n \rangle$ be a seq^n of +ve real numbers s.t. $|f_n(x)| \leq M_n$ $\forall\, x \in D$, $n \in \mathbb{N}$. If the series $\sum M_n$ is convergent, then $\sum f_n$ is uniformly convergent on D.

Proof :– Let $\epsilon > 0$ be given arbitrary real numbers

For $m > n$, $m, n \in \mathbb{N}$,

Consider $|f_{n+1}(x) + \cdots + f_m(x)|$

$$\leq |f_{n+1}(x) + f_{n+2}(x) + \cdots + f_m(x)|$$

$$\leq M_{n+1} + M_{n+2} + \cdots + M_m \quad \text{——} (1) \quad (\text{by using } |f_n(x)| \leq M_n$$

$\forall\, x \in D.$

If $\sum M_n$ is convergent, then by Cauchy's criteria of series of real no., we get that

for given $\epsilon > 0$, $\exists$ a natural no. $K(\epsilon)$ s.t.

$$|M_{n+1}+M_{n+2}+\cdots+M_m| < \epsilon \quad \forall\, m > n \geq K(\epsilon)$$

from (1)

$$|f_{n+1}(x)+f_{n+2}(x)+\cdots+f_m(x)| < \epsilon \quad \forall\, m > n \geq K(\epsilon), \ \forall\, x \in D$$

$\therefore$ By Cauchy's criteria for series of functions, we get that $\sum f_n$ is uniformly convergent on D.

Here proved

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Q. 1 Discuss the convergence and the uniform convergence of the series $\sum f_n$, where $f_n(x)$ is given by :—

[a] $(x^2+n^2)^{-1}$

Here, $|f_n(x)| = \left|\dfrac{1}{x^2+n^2}\right| \quad \forall\, x \in \mathbb{R}$

$$= \frac{1}{x^2+n^2} \quad \forall\, x \in \mathbb{R}$$

$$\leq \frac{1}{n^2} \quad \forall\, x \in \mathbb{R}$$

$\because x^2 \geq 0$

$n^2+x^2 \geq n^2$

$\Rightarrow \dfrac{1}{n^2+x^2} \leq \dfrac{1}{n^2}$

let $M_n = \frac{1}{n^2}$ $\forall n \in \mathbb{N}$

Then, $|f_n(x)| \le M_n$ $\forall n \in \mathbb{N}$ & $x \in \mathbb{R}$ —(1)

and $\sum M_n = \sum \frac{1}{n^2}$ is a convergent series by p-test.

So, by weierstrass M-test, it follows that the series

$\sum \frac{1}{x^2+n^2}$ is uniformly convergent on $\mathbb{R}$.

$\boxed{b}$ $(nx)^{-2}$; $(x \neq 0)$

Here $f_n(x) = \frac{1}{n^2 x^2}$; $x \neq 0$

Now, the series

$$\sum f_n = \sum \frac{1}{n^2 x^2} = \frac{1}{x^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is Converged } \forall x \neq 0$$

(~~other~~ Since, the series $\sum \frac{1}{n^2}$ is cgt by P-test)

$\Rightarrow$ $\sum f_n$ is pointwise cgt $\forall$ $x \neq 0$ & $x \in \mathbb{R}$

Now $\left|\frac{1}{n^2 x^2}\right|$ $\forall x \neq 0$, $n \in \mathbb{N}$

$= \frac{1}{n^2 x^2}$, $x \neq 0$, $n \in \mathbb{N}$

$\le \frac{1}{n^2 a^2}$ if $|x| \ge a$ for some $a > 0$

$\Rightarrow$ $|f_n(x)| \le \frac{1}{n^2 a^2}$ $\forall x$ s.t. $|x| \ge a$ $\forall n \in \mathbb{N}$.

where $a>0$ is any real no.

$\Rightarrow \quad |f_n(x)| \le M_n \quad \forall\, x \in [a,\infty) \cup (-\infty,-a] \quad \forall\, n \in \mathbb{N}$

where $\quad M_n$ is $\dfrac{1}{n^2 a^2} \quad \forall\, n \in \mathbb{N}$

Now, $\sum M_n$ is a cgt series of real no.

$\therefore$ By W. M. test, $\sum f_n$ is uniformly converget on the set $(-\infty,-a] \cup [a,\infty)$, where $a>0$ is arb. real no.

Now we will show that $\sum f_n$ is not uniformly cgt on $\mathbb{R}$. For this, it is sufficient to show that the seq^n $\langle f_n(x) \rangle$ is not uniformly cgt on $\mathbb{R}-\{0\}$

$\epsilon_0 = 1/2$

Let $n_k = k$ & $x_k = \dfrac{1}{k} \in \mathbb{R}-\{0\} \quad \forall\, k \in \mathbb{N}$

$$|f_{n_k}(x_k)| = \left| \frac{1}{k^2 \times \frac{1}{k^2}} \right| = 1 > \epsilon_0 = \frac{1}{2} \quad \forall\, k \in \mathbb{N}$$

$\Rightarrow \quad \langle f_n \rangle$ is not uniformly cgt. on $\mathbb{R}-\{0\}$

$\Rightarrow \quad \sum f_n$ is not uniformly cgt on $\mathbb{R}-\{0\}$

$\boxed{c}$ $\quad \sin\left(\dfrac{x}{n^2}\right)$

Here $\quad f_n(x) = \sin\left(\dfrac{x}{n^2}\right) \quad \forall\, x \in \mathbb{R} \quad \forall\, n \in \mathbb{N}$

Now, $\quad |f_n(x)| = \left|\sin\left(\dfrac{x}{n^2}\right)\right| \quad \forall\, x \in \mathbb{R}$ & $\forall\, n \in \mathbb{N}$

$\le \left|\dfrac{x}{n^2}\right| \quad \forall\, x \in \mathbb{R}, \; \forall\, n \in \mathbb{N}$

$$= \frac{|x|}{n^2}$$

Let $M_n = \frac{|x|}{n^2}$

$\sum M_n = |x| \sum_{n=1}^{\infty} \frac{1}{n^2}$ is cgt by p-test

$\Rightarrow \sum f_n$ is absolutely cgt $\forall x \in \mathbb{R}$

$\Rightarrow \sum f_n$ is pointwise cgt on $\mathbb{R}$

Now,

$|f_n(x)| \leq \frac{|x|}{n^2} \quad \forall x \in \mathbb{R}, \ \forall n \in \mathbb{N}$

$\leq \frac{a}{n^2} \quad \forall x \in \mathbb{R}$ s.t. $|x| \leq a \ (a > 0)$

$\forall n \in \mathbb{N}$

$\Rightarrow \sum f_n$ is uniformly cgt on $[-a, a]$, where $a > 0$ is any real no.

Now, we will show that $\sum f_n$ is not uniformly cgt on $\mathbb{R}$

It is sufficient to show that $\langle f_n(x) \rangle$ is not uniformly cgt on $\mathbb{R}$.

Let $n_k = k$ and $x_k = \frac{\pi}{2} k^2$, $\epsilon_0 = 1/2$

then $|f_{n_k}(x_k)| = \left| \sin\left(\frac{\pi}{2} \cdot \frac{k^2}{k^2}\right) \right| = \left| \sin \frac{\pi}{2} \right| = 1 > \frac{1}{2} = \epsilon_0$

$\Rightarrow \langle f_n(x) \rangle$ is not uniformly cgt on $\mathbb{R}$

Hence, $\sum f_n(x)$ is not uniformly cgt on $\mathbb{R}$

$\boxed{d}$ $(x^n+1)^{-1}$; $x \neq 0$

Here, $f_n(x) = \dfrac{1}{x^n+1}$ for $x>0$

Case-I $0 < x < 1$

Then $\lim_{n\to\infty} x^n = 0$

& $\lim_{n\to\infty} f_n(x) = \dfrac{1}{x^n+1} = \dfrac{1}{1+0} = 1 \neq 0$

$\Rightarrow$ $\sum f_n(x)$ is not cgt, for $x \in \mathbb{R}$ s.t. $|x|<1$

Case-II If $x=1$, then $f_n(x) = (1+1)^{-1} = \frac{1}{2}$

$\Rightarrow$ $\lim_{n\to\infty} f_n(x) = \frac{1}{2} \neq 0$

$\Rightarrow$ $\sum f_n$ is not cgt at $x=1$

Case-III if $1 < x < \infty$, then

$|f_n(x)| = \dfrac{1}{x^n+1}$ $\forall\ x>1$, $n \in \mathbb{N}$

$< \dfrac{1}{x^n}$ $\forall\ x>1$ $\forall\ n \in \mathbb{N}$

$= \left(\dfrac{1}{x}\right)^n$

$\Rightarrow$ $\sum\left(\frac{1}{x}\right)^n$ is a geometric series with common ratio $\frac{1}{x}$

$\Rightarrow$ $\sum\left(\frac{1}{x}\right)^n$ is cgt.

so, by comparison test . $\sum|f_n(x)|$ is cgt

$\Rightarrow$ $\sum f_n(x)$ is cgt pointwise for $x>1$

Now, we will show that $\sum f_n$ is not uniformly cgt on $(1,\infty)$

let $n_k = k$ & $x_k = (2)^{1/k}$ $\in (1,\infty)$. $k \in \mathbb{N}$, $\epsilon_0 = 1/4$

then $|f_{n_k}(x_k)| = \left| \dfrac{1}{(2^{1/k})^k + 1} \right|$ $\forall\, k \in \mathbb{N}$

$= \dfrac{1}{2+1} = \frac{1}{3} > \frac{1}{4} = \epsilon_0$ $\forall k$

$\Rightarrow$ $\langle f_n(x) \rangle$ is not uniformly convergent on $(1,\infty)$

Hence

$\sum f_n$ is not uniformly cgt on $(1,\infty)$

$\boxed{e}$

$$f_n(x) = \frac{x^n}{x^n+1} \quad , \quad x \geq 0$$

Case-I For $0 \leq x < 1$

We know $\dfrac{x^n}{x^n+1} \leq x^n$ $\forall\, x \geq 0$, $\forall\, n \in \mathbb{N}$

$x > 0$
$x^n > 0$
$1+x^n > 1$
$\dfrac{1}{1+x^n} < 1$
$\dfrac{x^n}{1+x^n} < x^n$

& $\sum(x)^n$ is a geometric series with common ratio x & $0 \leq x < 1$

$\Rightarrow$ $\sum(x)^n$ is cgt

$\therefore$ By comparision test, $\sum f_n$ is cgt.

Case-II for $x=1$

$f_n(x) = \frac{1}{2} \neq 0$

$\Rightarrow \sum f_n$ is not uniformly cgt. on $x=1$

Case-III $x>1$

$$f_n(x) = \frac{1}{\left(\frac{1}{x}\right)^n + 1}$$

$\Rightarrow \lim f_n(x) = 1 \neq 0 \Rightarrow \sum f_n$ is not cgt for $x>1$

Hence, the series $\sum f_n$ is pointwise cgt on $[0,1)$

Now $|f_n(x)| = \frac{x^n}{x^n+1} \leq x^n \leq a^n$ for $0 \leq x \leq a$, $0 \leq a < 1$

Let $M_n = a^n$, $0 \leq a < 1$

then $|f_n(x)| \leq a^n$

& M_n is a geometric series

$\Rightarrow \sum M_n$ is cgt

So by "W. M test" $\sum f_n$ is uniformly cgt on $[0, a]$
where $0 \leq a < 1$

Now, we will show that $\sum f_n$ is not uniformly cgt on $[0,1)$

Let $n_k = k$ & $x_k = \left(\frac{1}{2}\right)^{k} \in [0,1)$

then

$$|f_{n_k}(x_k)| = \left| \frac{\left(\left(\frac{1}{2}\right)^{1/k}\right)^k}{\left(\left(\frac{1}{2}\right)^{1/k}\right)^k + 1} \right| \quad \forall k$$

$$= \left| \frac{\frac{1}{2}}{3/2} \right| = \frac{1}{3} > \frac{1}{4} = \epsilon_0$$

$\Rightarrow$ $\langle f_n \rangle$ is not uniformly cgt on ~~[0,1]~~ $[0,1)$

$\Rightarrow$ $\sum f_n$ is not uniformly cgt on $[0,1)$

—— x —— x ——

"Power Series"

- Power Series is a special kind of series of functions.

Let $\sum f_n$ be a series of function and $c \in \mathbb{R}$, $a_n \in \mathbb{R}$

Take
$$f_n(x) = a_n(x-c)^n$$

Then
$$\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \quad \text{--- (1)}$$

So (1) is called power series about the point $x=c$

for convenience, we take $c=0$

then
$$\sum_{n=0}^{\infty} f_n(x) = \sum a_n x^n \quad \text{--- (2)}$$

or take $x-c = x'$, then

$$\sum_{n=0}^{\infty} f_n(x) = \sum a_n x'^n$$ is the power series.

Eg-1 (2) can be written as

$$\sum a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

If $x=0$, then power series

$\sum a_n x^n = a_0$ is always convergent and its sum

is equal to a_0.

Eg-2

If we have

$$\sum n!\, x^n$$

Here $a_n = n!$

$l = \lim \left| \frac{a_{n+1}}{a_n} \right|$

if $l < 1$, Convergent
if $l = 1$, fails
if $l > 1$, Divergent

Using, ratio test

$$l = \lim \left| \frac{y_{n+1}}{y_n} \right| = \lim \left| \frac{(n+1)!\, x^{n+1}}{n!\, x^n} \right|$$

$$= \lim \left| (n+1)\, x^{(n+1)-n} \right|$$

$$= \lim_{n \to \infty} (n+1)\, |x|$$

$$= \infty$$

$\Rightarrow l = \infty > 1$

thus, $\sum n!\, x^n$ is divergent everywhere except 0
"for every x"

or

{0} is the only ~~set~~ set for the power series is convergent

EX-3 $\sum x^n$

then $l = \lim \left| \frac{y_{n+1}}{y_n} \right| = \lim \left| \frac{x^{n+1}}{x^n} \right| = \lim_{n \to \infty} |x| = |x|$

so, wherever

$l = |x| < 1 \rightarrow \sum x^n$ is convergent

i.e. for $(-1, 1)$, $\sum x^n$ is convergent.

At $x=-1$

the $\sum x^n = (-1)+1-1+1-1+1 \cdots$

then

$$S_n = \left\{ \begin{array}{ll} -1, & \text{for } n = \text{odd} \\ 0, & \text{for } n = \text{even} \end{array} \right\} \rightarrow \text{oscillatory}$$

For $x=1$,

$$\sum x^n = \underbrace{1+1+\cdots+1}_{n\text{-times}} = n$$

Thus for $x=1 \rightarrow$ divergent

$\therefore$ Set of convergence $= (-1,1)$

$= \{x \in \mathbb{R} : |x| < 1\}$.

<u>eg-4</u> $\sum \frac{x^n}{n!}$.

$$l = \lim \left| \frac{y_{n+1}}{y_n} \right| = \lim \left| \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n} \right|$$

$$= \lim \frac{|x|}{n+1}$$

$$= 0 < 1$$

thus

$\sum \frac{x^n}{n!}$ is convergent $\forall x \in \mathbb{R}$

* The Power series for which $\{0\}$ is the only set of Convergence., such series are called no where convergent.

* Power series which are convergent on $\mathbb{R}$ or $\forall x \in \mathbb{R}$; such series are called everywhere converget.

* In case of Power series, limit comes out to be either '0' or ∞ or dependent on $|x|$.

The case, where limit comes out to be dependent on $|x|$

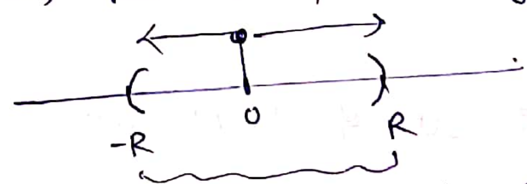
For example :-

take $\sum (4x)^n$

then for convergence $|4x| < 1$

$\Rightarrow |x| < \frac{1}{4}$

In general, $|x| < \boxed{R} \rightarrow$ Radius of convergence



The distance for which Power series remains convergent.

For radius of Convergence R :—

$$R = \frac{1}{\lim \sup |a_n|^{1/n}}$$ where $\langle a_n \rangle$ is bdd.

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Take $\beta = \limsup |a_n|^{1/n}$

So • $\beta = +\infty$, $R = 0$ finite no.

• If $\langle a_n \rangle$ is unbounded, then $\beta = +\infty$

If $\langle a_n \rangle$ is bdd, then β is either 0 or some non zero no.

So, for example-2 $R = 0$

For example-3 $R = 1$ $(\because |x| < 1)$

for example-4 series is every where convergent
$\therefore R = +\infty$.

* For $x \neq 0$

Power Series

Convergent Divergent

The set of points for which the power series is convergent is known as interval of convergence.

* Limit superior & limit inferior for a bounded funct^n:

We define

$$v_N = \sup\{x_n : n \geq N\}.$$

$$\text{|||} \quad u_N = \inf\{x_n : n \geq N\}.$$

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Then

$$\limsup x_n = \lim x_n = \lim_{N\to\infty} v_N$$

||| $$\liminf x_n = \lim x_n = \lim_{N\to\infty} u_N$$

* **Ex-1** $x_n = (-1)^n$

then $\langle x_n \rangle = \{-1, 1, -1, 1, \dots\}$

If $N=1$

$u_1 = -1$

$v_1 = 1$

If $N=2$

$u_2 = -1$

$v_2 = 1$

$\therefore$ We have $u_N = -1 \quad \forall\, N \in \mathbb{N}$

$v_N = 1 \quad \forall\, N \in \mathbb{N}$

Then $\limsup x_n = \lim_{N\to\infty} v_N = 1$

& $\liminf x_n = \lim_{N\to\infty} u_N = -1$

But $\limsup x_n \neq \liminf x_n$

$\therefore \langle x_n \rangle$ is oscillatory

* **EX-2 :-** $x_n = \frac{1}{n}$, then

$\langle x_n \rangle = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}\}$

If $N = 1$, then

$u_1 = 0$

$v_1 = 1$

$N = 2$, then

$u_2 = 0$

$v_2 = \frac{1}{2}$

If $N = 3$, then

$u_3 = 0$

$v_3 = 1/3$

$\therefore$ In general

$$\left.\begin{aligned} u_N &= 0 \\ v_N &= \frac{1}{N} \end{aligned}\right\} \forall\ N \in \mathbb{N}$$

$\liminf x_n = 0$

& $\limsup x_n = \lim_{N \to \infty} \frac{1}{N} = 0$

In this case $\limsup x_n = \liminf x_n = \lim$ of seq^n $\langle x_n \rangle$

* Note :—

If $\langle x_n \rangle$ is bdd from below but not from above

then take $\limsup x_n = +\infty$

||| If $\langle x_n \rangle$ is unbdd from below and bdd from above

then $\lim x_n = -\infty$

* Remark :— Limit superior & limit inferior always exist irrespectively of existence of limit of a seq^n.

Imp

Cauchy Hadamard Theorem (Thm 23·1) :—

@...

For the power series $\sum a_n x^n$, let

$$\beta = \limsup |a_n|^{1/n}$$

and $R = \frac{1}{\beta}$, (if $\beta = 0$, we ~~say~~ take $R = +\infty$ & if $\beta = +\infty$, we take $R = 0$)

Then

(i) the power series converges for $|x| < R$

(ii) the power series diverges for $|x| > R$

Proof :— Here, we will apply nth root test to the series $\sum a_n x^n$

For $x \in \mathbb{R}$, let

$$\alpha_x = \limsup |a_n x^n|^{1/n}$$

$$= \lim_{n \to \infty} \sup |a_n|^{1/n} \cdot |x|$$

$$= |x| \cdot \limsup |a_n|^{1/n}$$

$$= |x| \cdot \beta$$

$$= \frac{|x|}{R}$$

Case - I when $0 < R < \infty$ or $0 < \beta < \infty$, then by nth root test, if

$$\alpha_x = \frac{|x|}{R} < 1$$

i.e. $|x| < R$, then Power series is convergent

& if
$$\alpha_x = \frac{|x|}{R} > 1$$

i.e. $|x| > R$, then Power series diverges.

Case-II When $R = +\infty$, i.e. $\beta = 0$

Then $\alpha_x = 0$, for all x

$\Rightarrow \alpha_x = 0 < 1$

$\Rightarrow \alpha_x < 1$

$\therefore$ By n^{th} root test $\sum a_n x^n$ converges for all x
(i.e. $\forall x$ s.t. $|x| < \infty = R$)

Case-III If $R = 0$, i.e. $\beta = +\infty$

Then $\alpha_x = +\infty$ for all $x \neq 0$
> 1

$\therefore$ By n^{th} root test, $\sum a_n x^n$ diverges for all $x \neq 0$
(i.e. for all x s.t. $|x| > 0 = R$)

Hence Proved.

Remark :— If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, then it is equal to β

$$\boxed{\begin{gathered}\text{If } \lim |a_n|^{1/n} \text{ exists, then} \\ \limsup |a_n|^{1/n} = \lim |a_n|^{1/n} = \lim \left|\frac{a_{n+1}}{a_n}\right|\end{gathered}}$$

$\therefore \beta = \frac{1}{R}$ i.e. $R = \frac{1}{\beta} = \lim \left|\frac{a_n}{a_{n+1}}\right|$ #

Example-1 $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

Sol^n:- Here $R = \frac{1}{\beta} = \lim \left| \frac{a_n}{a_{n+1}} \right|$ $\left(\because a_n = \frac{1}{n!}\right)$

$= \lim \left| \frac{1}{n!} \times (n+1)! \right|$

$= \lim_{n\to\infty} (n+1)$

$= \infty$

$\Rightarrow \sum \frac{1}{n!} x_n$ converges for all x s.t. $|x| < \infty$

i.e. for all real no.

$\therefore$ Interval of Convergence $= \mathbb{R}$ {set of real no.'s}

$\Rightarrow$ It is an everywhere convergent series.

Example-2 $\sum_{n=0}^{\infty} x^n$

Here $a_n = 1$

$\Rightarrow R = 1$

at $x=1$, $\sum x^n = 1+1+\cdots\cdots$, i.e. diverges

When $x = -1$

then $\sum x^n = 1-1+1-1+1-\cdots$

& $S_n = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$

$\Rightarrow$ the seq^n of Partial sum is oscillating,
thus the power series is not convergent at $x=-1$

$\therefore$ Interval of convergence $= (-1, 1)$

Example-3 $\quad \sum_{n=1}^{\infty} \frac{1}{n} x^n$

Here $a_n = \frac{1}{n}$

$$R = \frac{1}{\beta} = \lim \left| \frac{a_n}{a_{n+1}} \right| = \lim \left| \frac{n}{n+1} \right| = \lim_{n \to \infty} \left| \frac{1}{1+\frac{1}{n}} \right| = 1$$

OR $\quad \beta = \lim \sup \left| \frac{1}{n} \right|^{1/n}$

$$= \lim \sup \frac{1}{|n|^{1/n}}$$

$$= 1 \qquad \left(\because n^{1/n} \to 1\right)$$

Then $\quad R = \frac{1}{\beta} = \frac{1}{1} = 1$

At $x=1$

Then $\sum \frac{1}{n} x^n = \sum_{n=1}^{\infty} \frac{1}{n}$, which is divergent

when $x=-1$

then $\sum \frac{1}{n} x^n = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} + - -$

which is convergent by lebnitz test.

$$\left\{ \text{lebnitz test:- } a_{n+1} \le a_n \Rightarrow \frac{1}{n+1} < \frac{1}{n} \quad \& \quad \lim |a_n| = 0 \Rightarrow \lim \frac{1}{n} \to 0 \right\}.$$

Therefore, Interval of Convergence $= [-1, 1)$

Example-4 :-

$$\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$$

Here $R = \frac{1}{\beta} = \lim \left| \frac{a_n}{a_{n+1}} \right|$

$$= \lim \left| \frac{n^2}{(n+1)^2} \right| = \lim \left| \frac{n^2}{n^2+2n+1} \right| = 1$$

When $x = 1$

$\sum \frac{1}{n^2} x^n$ converges by p-test

When $x = -1$

$$\sum \frac{1}{n^2} x^n = \sum \frac{(-1)^n}{n^2}$$

Here $\frac{1}{(n+1)^2} < \frac{1}{n^2}$

$\&$ $\lim \left(\frac{1}{n^2} \right) = 0$

$\therefore$ By lebnitz test, Power series converges

Thus, Interval of Convergence $= [-1, 1]$

EX - 5 Do yourself

Example - 6 :-

$$\sum_{n=0}^{\infty} 2^{-n} x^{3n} \quad ; \quad R = ?$$

Interval of convergence :-

$$R = \frac{1}{\limsup |a_n|^{1/n}} = \lim \left| \frac{a_n}{a_{n+1}} \right|$$

let $a_n = 2^{-n} = \frac{1}{2^n}$, $x^3 = y$

$$\Rightarrow \quad R = \lim \left| \frac{2^{-n}}{2^{-(n+1)}} \right| = \lim \left| \frac{2^{n+1}}{2^n} \right| = |2| = 2$$

i.e.

Power Series $\sum_{n=0}^{\infty} 2^{-n} y^n$ Converges if $|y| < 2$

i.e. $|x^3| < 2 \Rightarrow |x| < 2^{1/3}$

i.e. $\sum_{n=0}^{\infty} 2^{-n} x^{3n}$ converges if $|x| \; 2^{1/3}$

$\therefore$ Interval of convergence $= (-2^{1/3}, 2^{1/3})$

Example - 7 :- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n$

Put $x - 1 = y$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot y^n$$

Here $a_n = \frac{(-1)^{n+1}}{n}$

$$R = \lim \left| \frac{a_n}{a_{n+1}} \right| = 1$$

$\therefore$ the Power series $\sum \frac{(-1)^{n+1}}{n} y^n$ converges if $|y| < 1$

the Power series $\sum \frac{(-1)^{n+1}}{n} \cdot (x-1)^n$ converges if

$$|x-1| < 1$$

$$\Rightarrow -1 < x-1 < 1$$

i.e. $0 < x < 2$ $\rightarrow$ (interval of convergence)

At end points

when $x=0$, the Power series becomes

$$\sum \frac{(-1)^{n+1}(-1)^n}{n} = \sum \frac{(-1)}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges by p-test

when $x=2$, the power series becomes

$$\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

which is cgt by Libnitz test.
($\because$ it is decreasing seq^n and n^{th} term tends to 0)

So the interval of Convergence for given Power series is $(0, 2]$

P-176 Exercises-23.1

Q.1 (a to e) Do yourself.

Theorem: 26.1 Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R>0$, {Possibly $R=+\infty$}. If $0<R_1<R$, then the power series converges uniformly on $[-R_1, R_1]$ to a continuous function.

Proof:– Let $0<R_1<R$

By defn of radius of convergence, the power series $\sum a_n x^n$ and $\sum |a_n| x^n$ will have same radius of convergence.

$$\left\{ \begin{array}{l} \because R \text{ is defined in term of } |a_n| \\ \text{i.e. } R = \dfrac{1}{\lim\sup |a_n|^{1/n}} \end{array} \right\}$$

$\therefore$ By Cauchy Hadamard theorem, the $\sum |a_n| x^n$ converges if $|x|<R$

$\Rightarrow$ $\sum |a_n| x^n$ is convergent at $x = R_1$

$$\left(\begin{array}{l} \because 0<R_1<R \\ \text{take } x = R_1 \\ \Rightarrow R_1 < R \end{array} \right)$$

$\Rightarrow \sum_{n=0}^{\infty} |a_n| R_1^n$ is convergent — ①

Consider, $|a_n x^n|$

$= |a_n||x^n|$

$= |a_n||x|^n$

$\leq |a_n| R_1^n \quad \forall\, x \in [-R_1, R_1] \quad \forall\, n \in \mathbb{N}$

i.e. $|x| \leq R_1$

thus, we get $|a_n x^n| \leq |a_n| |x|^n \leq |a_n| R_1^n \quad \forall\, n \in \mathbb{N}$ — ②

from ① and ② and using wierstrass M-test, we get $\sum a_n x^n$ is uniformly convergent on $[-R_1, R_1]$

$\left\{ \because \text{Take } |f_n(x)| = |a_n x^n| \ \& \ M_n = |a_n| R_1^n \right.$

$\Rightarrow |f_n(x)| \leq M_n \quad \forall\, n \in \mathbb{N}$

$\& \ M_n$ Converges by ① $\}$

Now, Since each $a_n x^n$ is continuous on $[-R_1, R_1]$, and (being a Polynomial)

$\sum a_n x^n$ is uniformly continuous on $[-R_1, R_1]$, therefore by result " Interchange of limit & continuity ", the limit function or sum function is continuous on $[-R_1, R_1]$

Hence proved

Corollary 26.2 :- The power series $\sum a_n x^n$ converges to a continuous function on $(-R, R)$.

Proof :- If $x_0 \in (-R_1, R)$, then $x_0 \in [-R_1, R_1]$ for some $0 < R_1 < R$

Therefore, by above theorem, the limit or sum function of the power series is continuous on $[-R_1, R_1]$

In, Particular, It is continuous at $x = x_0$

$\because$ $x_0 \in (-R_1, R)$ was arbitrary, therefore the limit function is continuous on $(-R, R)$.

$\#$

Results

$$\left(\sum_{n=0}^{\infty} a_n x_n\right)' = \sum (a_n x^n)'$$

$$= \sum_{n=1}^{\infty} a_n \cdot n x^{n-1}$$

$\because \sum a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

$(\sum a_n x_n)' = a_1 + a_2(2x) + a_3(3x^2) + \cdots$

If we diff $\sum_{n=0} a_n x^n$ once, then $\sum a_n x^n$ start from $n=1$

twice, then " " $n = 2$

II) $$\int \left(\sum_{n=0}^{\infty} a_n x^n\right) dx = \sum_{n=0}^{\infty} \int a_n x^n \, dx$$

$$= \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$$

i.e. whether we integrate $\sum a_n x^n$ once or twice, the $\int (\sum a_n x^n) dx$ starts with $n=0$ always

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Lemma 26.3 :-

If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then the power series

$$\sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

will have the same radius of convergence R.

Proof :-

We observe that the power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ and $\sum n a_n x^n$ will have same radius of convergence because the second series is x times the first series.

i.e.

$$\sum n a_n x^n = x \times \left[\sum n a_n x^{n-1} \right]$$

(similarly)
/// the series $\sum \frac{a_n x^{n+1}}{n+1}$ and $\sum \frac{a_n x^n}{n+1}$ will have same radius of convergence because the second series is $1/x$ times the first series.

i.e.

$$\sum \frac{a_n x^n}{n+1} = \frac{1}{x} \times \left(\sum \frac{a_n x^{n+1}}{n+1} \right) \quad ; \quad x \neq 0$$

Now, for $\sum_{n=1}^{\infty} n a_n x^n$, radius of convergence

$$= \frac{1}{\limsup |n a_n|^{1/n}}$$

$$= \frac{1}{\limsup |n|^{1/n} |a_n|^{1/n}}$$

$\because \lim |n|^{1/n} = 1$

$\limsup |n|^{1/n} = 1$

$$= \frac{1}{\limsup |a_n|^{1/n}}$$

$$= \frac{1}{\beta} = R$$

Now, for $\sum_{n=0}^{\infty} \frac{a_n x^n}{n+1}$, the radius of convergence

$$= \frac{1}{\limsup \left| \frac{a_n}{n+1} \right|^{1/n}}$$

$$= \frac{1}{\left(\frac{\limsup |a_n|^{1/n}}{\limsup |n+1|^{1/n}} \right)}$$

$$= \frac{1}{\limsup |a_n|^{1/n}}$$

$\because \lim_{n \to \infty} \left(\frac{1}{n+1} \right)^{1/n} = 1$

$\therefore \limsup |n+1|^{1/n} = 1$

$$= R$$

Thus $\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ and $\sum_{n=1}^{\infty} n a_n x^{n-1}$ have the same radius of convergence as $\sum_{n=0}^{\infty} a_n x^n$.

Theorem 26.4 :- Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \quad \text{for } |x| < R \quad \text{--- (1)}$$

Proof :-

Proof :- Fix any x and let $x<0$

Consider the interval $[x,0]$

By theorem 26.1, the power series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly to f on $[x,0]$.

$\Rightarrow$ The seq^n of partial sum $\langle \sum_{K=0}^{n} a_K x_K \rangle$ converges uniformly to f on $[x,0]$, therefore, by thm 8.2.4,

We get
$$\int_x^0 f(t)\,dt = \lim_{n\to\infty} \int_x^0 \sum_{K=0}^{n} a_K t^K\,dt$$

$$= \lim_{n\to\infty}\left(\sum_{K=0}^{n} a_K \left(\int_{x}^{0} t^K dt\right)\right) \quad (\because \text{ Sum is finite}$$

$$= \lim_{n\to\infty}\left(-\sum_{K=0}^{n} a_K \frac{x^{K+1}}{K+1}\right)$$

$$= -\sum_{K=0}^{\infty} a_K \frac{x^{K+1}}{K+1}$$

(using lemma 26.3
$\sum a_n x^n$ & $\sum \frac{a_n x^{n+1}}{n+1}$
have same R

$\because$ x is arbitrary, therefore

$$\int_x^0 f(t)\,dt = -\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}, \quad \text{for } x\in(-R,R) \text{ where } x<0$$

$$\Rightarrow \int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{(n+1)} \quad \text{for } x\in(-R,R) \text{ where } x<0$$

Similarly, we can prove the case for $x>0$

thus, we get $\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ for $|x|<R$. #

Theorem 26.5 :-

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R > 0$.

Then, f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{for } |x| < R$$

Proof :- By lemma 26.3, the power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} a_n \cdot n x^{n-1}$ have the same radius of convergence R.

let $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for $|x| < R$

By thm 26.4, we can integrate it term by term

i.e. $\int_0^x g(t)\, dt = \sum_{n=1}^{\infty} a_n x^n$

$$= f(x) - a_0$$

$$\Rightarrow \quad f(x) = \int_0^x g(t)\, dt + a_0$$

If $0 < R_1 < R$, then

$$f(x) = \int_{-R_1}^{x} g(t)\, dt + a_0 - \int_{-R_1}^{0} g(t)\, dt$$

$$f(x) = \int_{-R_1}^{x} g(t)\, dt + K \quad \text{for } |x| < R_1 \quad ; \; K = a_0 - \int_{-R_1}^{0} g(t)\, dt$$

$\because$ g is continuous on $[-R_1, R]$.

therefore

By fundamental theorem of Calculus - II, we get

f is differential and

$$f'(x) = g(x) \quad \text{for } |x| < R_1 < R$$

i.e. $$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{for } |x| < R$$

Hence proved.

Imp

Example -1 To find the expansion of $\log(1-x)$.

Sol^n :- Consider $$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$= 1 + x + x^2 + x^3 + \dots \quad \text{(By binomial thm)}$$

$$= \sum_{n=0}^{\infty} x^n \quad ; \quad |x| < 1$$

Here, $a_n = 1$

$$R = \frac{1}{\limsup |a_n|^{1/n}} = 1$$

As $$\sum_{n=0}^{\infty} x^n \quad ; \quad |x| < 1$$

Partial sum of this series is not convergent at end Points i.e. $x = 1, -1$

So using thm 26.4, we can integrate the partial sum term by term, we get

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad ; \quad |x|<1$$

$$\Rightarrow \int_0^x \frac{dt}{1-t} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad , \quad |x|<1$$

$$\Rightarrow -\log(1-t)\Big|_0^x = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad , \quad |x|<1$$

$$\Rightarrow -\log(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad ; \quad |x|<1$$

$$\Rightarrow \log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$$

which is divergent at $x=1$. but convergent for $x=-1$ so $x \in [-1,1)$

$$\Rightarrow \boxed{\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}}$$

Interval of Convergence $= [-1, 1)$

* Expansion of $\log(1+x)$ can be obtained. by replacing x by $-x$ & then we get

$$\log(1+x) = -\sum_{n=1}^{n} \frac{(-1)^n x^n}{n} \quad — ① \quad |x|<1$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

when $x = 1$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x)^n}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$$

which converges - (by libnitz test)

For $x = -1$

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = -1 - \frac{1}{2} - \frac{1}{3} \cdots\cdots$$

$$= -\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\cdots\right)$$

(divergent by p-test)

$\therefore$ Interval of convergece of

$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ is $[-1, 1]$.

Hence

$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$ ($\because$ series is cgt at $x=1$)

(only statement)

★ **Theorem 26.6 (Abel's Theorem) :-**

let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with finite positive radius of Convergence R. If the series converges at $x = R$, then f is continuous at $x = R$. If the series converges at $x = -R$ then f is continuous at $x = -R$.

Example :- 2

(Exp-1) From ① we know that

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = f(x)$$

for $x = 1$, this series converges by Leibnitz test.

$\therefore$ By Abel's theorem, the series represents a function f on $(-1, 1]$, which is continuous at $x=1$

Also, the function $\log(1+x)$ is continuous at $x=1$

$\therefore$ Both the functions ($\log(1+x)$ & f) agree at $x=1$

i.e. $\log(1+1) = f(1)$

$\because$ if $\langle x_n \rangle$ is any seq^n in $(-1, 1)$, converging to 1

$\because$ $\log(1+x)$ is cts

$\therefore$ $\log 2 = \lim \log(1+x_n)$

$= \lim f(x_n)$

$\log 2 = f(1)$

i.e. $\boxed{\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots}$

Exercise

Q. 26.8 Show that

(a) $$\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2} \quad \text{for } x \in (-1, 1)$$

(b) $$\tan^{-1} x = \arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)}, \quad |x| < 1$$

(c) Show that equality holds in part (b) for $x = 1$
use this to find a formula for π

(d) What happens for $x = -1$?

Soln :-

(a) Using binomial expansion, we get

$$\frac{1}{1+x^2} = (1+x^2)^{-1}$$

Put $x^2 = y$

$$= (1+y)^{-1}$$

$$= 1 - y + y^2 - y^3 + y^4 \ldots\ldots = \sum_{n=0}^{\infty} (-1)^n y^n, \quad |y| < 1$$

$$= 1 - x^2 + x^4 - x^6 + x^8 \ldots\ldots, \quad |x^2| < 1$$

i.e. $|x| < 1$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n}, \quad |x| < 1$$

Thus, we get $$\sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2} \; ; \quad |x| < 1$$ #

(b)

Integrating the power series term by term, we get (By thm 26.5 (state it))

$$\int_0^x \frac{1}{1+t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} \; ; \; |x|<1$$

$$\Rightarrow \tan^{-1} t \Big|_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\Rightarrow \tan^{-1} x - \tan^{-1} 0 = \sum (-1)^n \frac{x^{2n+1}}{2n+1}$$

Thus,

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)} \; ; \quad |x|<1$$

$\boxed{C}$ At $x=1$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot (1)^{2n+1}}{(2n+1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots$$

which converges by Leibnitz test.

$\therefore$ By Abel's theorem

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}$$ represents a continuous function.

which is continuous at $x = 1$

Also $\tan^{-1}x$ is continuous at $x = 1$

Thus $\tan^{-1}(1) = f(1)$

{ Let $\langle x_n \rangle$ be a seq^n in $(-1,1)$ converging to 1

Then $f(1) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \tan^{-1}(x_n) = \tan^{-1}(1)$ }.

$$\Rightarrow \frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Imp

$$\Rightarrow \boxed{\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)}$$

Impt

$\boxed{d}$ Put $x = -1$ in the power series, we get

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1}}{(2n+1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n} (-1)}{(2n+1)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)}$$

$$= -1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} \cdots$$

$$= -\left\{ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \right\}$$ which converges by leibnitz test.

Therefore by Abel's theorem, equality in part (b) holds for $x = -1$

i.e.

$$\tan^{-1}(-1) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)}$$

$$\Rightarrow -\frac{\pi}{4} = -\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

$$\Rightarrow \pi = 4\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

Hence, solved

Q. 26.2 (a) Observe that

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2} \quad \text{for } |x| < 1$$

(b) Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$

(c) Evaluate $\sum_{n=1}^{\infty} \frac{n}{3^n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{3^n}$

Sol^n:- (a)

We know

$$\frac{1}{1-x} = (1-x)^{-1}$$

$$= 1 + x + x^2 + \cdots$$

$$= \sum_{n=0}^{\infty} x^n \quad ; \quad |x| < 1$$

Now, differentiating, it term by (thm 26.4), we get

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad ; \quad |x| < 1$$

Multiply both sides by x, we get

$$\boxed{\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n \quad ; \quad |x| < 1}$$

(b) $$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^n} = \frac{1}{2} \times \frac{1}{1/4} = 2$$

(c) $$\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1/3}{(1-1/3)^2} = \frac{1}{3} \times \frac{9}{4} = 3/4$$

$$\sum_{n=1}^{\infty} \frac{n(-1)^n}{3^n} = \frac{-1/3}{(4/3)^2} = -\frac{3}{16}$$

Q. 26.3 Use ques (26.2) to derive formula for

$$\sum_{n=1}^{\infty} n^2 x^n$$

Soln

We have

$$\sum_{n=1}^{\infty} x^n \cdot n = \frac{x}{(1-x)^2} \; ; \quad |x|<1 \quad \text{--- (1)}$$

Now differentiating (1) term by term (by thm 26.4)

we get

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{(1-x)^2 (1) + x(2(1-x))}{(1-x)^4}$$

$$= \frac{(1-x)^2 + 2x - 2x^2}{(1-x)^4}$$

$$= \frac{(1-x)(1-x) + 2x(1-x)}{(1-x)^4}$$

$$= \frac{(1-x)\{1-x+2x\}}{(1-x)^4}$$

$$= \frac{1+x}{(1-x)^3}$$

Multiply by x both sides

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3} \; , \quad |x|<1$$

(b) Calculate

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{n^2}{3^n}$$

Soln :-

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{\frac{1}{2}\left(\frac{3}{2}\right)}{\left(\frac{1}{2}\right)^4} \quad ; |x - \tfrac{1}{2}| < 1$$

$$= \frac{3}{4} \times 16 = 12$$

$|||$

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n} = \frac{\frac{1}{3}\left(\frac{4}{3}\right)}{\left(\frac{2}{3}\right)^4} = \frac{4}{9} \times \frac{81}{16} = \frac{9}{4}$$

Q: 26.5 :- Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $n \in \mathbb{N}$

Show that $f' = f$

diff it term by term, we get

$$f'(x) = \sum_{n=1}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

$$\left\{ \begin{aligned} \sum_{n=0}^{\infty} \frac{x^n}{n!} &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \\ \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \end{aligned} \right\} \quad \because e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Q. 26. 6

Let $S(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ ——— ①

$C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ ——— ②

(a) Show that $S'(x) = C(x)$ and $C'(x) = -S(x)$

(b) Show that $(S^2 + C^2)' = 0$

(c) Show that $S^2 + C^2 = 1$

Proof

(a) Differentiate ① term by term, we get

$$S'(x) = 1 - \frac{3x^2}{3 \times 2} + \frac{5x^4}{5 \times 4!} - \dots$$

$$S'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\Rightarrow \quad S'(x) = C(x)$$

||| Differentiate ② term by term, we get.

$$C'(x) = -\frac{2x}{2!} + \frac{4x^3}{4 \times 3!} - \frac{6x^5}{6 \times 5!} + \dots$$

$$C'(x) = -\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right]$$

$$\Rightarrow \quad \boxed{C'(x) = -S(x)}$$

[b] To Prove $(s^2+c^2)' = 0$

Take L.H.S. =

$$(s^2+c^2)' = (s^2)' + (c^2)'$$

$$= 2s\,s' + 2c\,c'$$

$$= 2s.c + 2c.(-s)$$

$$= 2sc - 2cs$$

$$= 0 \qquad \text{(By Part (a))}$$

$$= \text{RHS}.$$

Hence proved.

[c] To Prove $(s^2+c^2) = 1$

Proof. $\left((s(x))^2 + (c(x))^2\right)' = 0 \quad \forall\ x \in \mathbb{R}$

$\Rightarrow (s(x))^2 + (c(x))^2 = \text{Constant} \quad \forall\ x \in \mathbb{R}$

$\Rightarrow (s(x))^2 + (c(x))^2 = A \quad \forall\ x \in \mathbb{R}$

$$\left\{ \begin{aligned} \because\ s(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\ c(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\ \& \quad s(0) &= 0 \\ c(0) &= 1 \end{aligned} \right]$$

Take $x=0$, $(s(0))^2 + (c(0))^2 = A$

@

$\Rightarrow \quad 0+1 = A$

Thus, we get

$$(s(x))^2 + (c(x))^2 = 1 \qquad \forall x \in \mathbb{R}$$

$\Rightarrow \quad s^2 + c^2 = 1$

Hence proved.

* Remark :-

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

&

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Q. 26.7 let $f(x) = |x| \quad \forall x \in \mathbb{R}$

Is there a power series $\sum a_n x^n$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \forall x \; ??$$

Soln :- let if possible, a power series exists s.t.

$$|x| = \sum_{n=0}^{\infty} a_n x^n \qquad \forall x \in \mathbb{R}$$

Using thm (26.5), we can differentiate it term by term

but $|x|$ is not differentiable at $x = 0 \in \mathbb{R}$

which is a contradiction.

Thus $\nexists$ any power series $\sum_{n=0}^{\infty} a_n x^n$ s.t.

$$|x| = \sum_{n=0}^{\infty} a_n x^n \quad \forall \; x \in \mathbb{R}$$

Thm (27.5)

* Weierstrass's Approximation Theorem :-

Statement :- Every continuous function on $[a, b]$ can be uniformly approximated by polynomial on $[a,b]$.

In other words, given a continuous function f on $[a,b]$. $\exists$ a seq^n $\langle P_n \rangle$ of polynomials s.t. $\langle P_n \rangle$ converge to f uniformly on $[a,b]$ i.e.

$$\langle P_n \rangle \rightrightarrows f \text{ on } [a,b].$$

Q. 27.3 Show that $\nexists$ a seq^n of polynomial converging uniformly on $\mathbb{R}$ to f if

(a) $f(x) = \sin x$

(b) $f(x) = e^x$

Sol^n :- (a) $f(x) = \sin x$

Let, if possible, $\exists$ a seq^n $\langle P_n \rangle$ of polynomials s.t.

$$P_n \rightrightarrows f \text{ on } \mathbb{R}.$$

$\because$ $P_n \rightrightarrows f$ on $\mathbb{R}$

$\therefore$ By defⁿ, for each $\epsilon > 0$, $\exists$ a natural no. $K(\epsilon)$ s.t.

$$|P_n(x) - f(x)| < \epsilon \quad \forall\, n \geq K(\epsilon)$$
$$\forall\, x \in \mathbb{R}$$

$$\Rightarrow |P_n(x) - \sin x| < \epsilon \quad \forall\, n \geq K(\epsilon)$$
$$\forall\, x \in \mathbb{R}$$

$$\Rightarrow \quad P_n(x) < \sin x + \epsilon \quad \forall\, x \in \mathbb{R}, \ \forall\, n \geq K(\epsilon)$$

Here, since $\sin x$ is bdd, therefore we get

$P_n(x)$ is bounded for all $n \geq K(\epsilon)$ and $\forall\, x \in \mathbb{R}$

which is a contradiction.

$\because$ $P_n(x)$ is unbounded $\forall\, x \in \mathbb{R}$.

thus, $\nexists$ any seqⁿ $\langle P_n \rangle$ s.t.

$$\langle P_n \rangle \rightrightarrows f \text{ on } \mathbb{R}.$$

$\boxed{b}$

$$f(x) = e^x$$

let, if possible, $\exists$ a seqⁿ $\langle P_n \rangle$ of polynomial s.t.

$$\langle P_n \rangle \rightrightarrows e^x \ \forall\, x \in \mathbb{R}$$

$\therefore$ By defⁿ, for each $\epsilon > 0$, $\exists$ a natural no. $K(\epsilon)$ s.t.

~~$\langle P_n \rangle \rightrightarrows$~~

$|P_n(x) - f(x)| < \epsilon \quad \forall\ n \geq k(\epsilon)$
$\forall\ x \in R$

$\Rightarrow \quad |P_n(x) - e^x| < \epsilon \quad \forall\ n \geq k \ \ \& \ \forall\ x \in R$

$\Rightarrow \quad e^x - \epsilon < P_n(x) < e^x + \epsilon$

$\Rightarrow \quad e^x < P_n(x) + \epsilon \quad \forall\ n \geq k(\epsilon), \ \forall x \in \mathbb{R}$

$\Rightarrow \quad 1 < P_n(x)\, e^{-x} + \epsilon \cdot e^{-x} \quad \forall\ n \geq k(\epsilon)$
$\forall\ x \in \mathbb{R}$

Here, as $x \to \infty$,
R.H.S. $= (P_n(x)\, e^{-x} + \epsilon \cdot e^{-x}) \to 0$ which is contradiction

$\because\ 1 \not< 0$

thus, our assumption was wrong

Hence $\nexists$ any seq^n $\langle P_n \rangle$ s.t.

$\langle P_n \rangle \rightrightarrows f$ on $\mathbb{R}$

Hence Proved

—x—x—

Remark :- In ques (27.3), if we put a restriction on x, then $P_n(x)$ is bounded for all those x then there may exist a seq^n $\langle P_n \rangle$ s.t.

$P_n \rightrightarrows f$

"Convergence & Divergence of β & Γ - functions"

(Application Imp)

* Proposition 9.55 :— Let $a, b \in \mathbb{R}$ and $a < b$ and $f, g : (a, b] \rightarrow \mathbb{R}$ s.t. f and g are integrable on $[x, b]$ $\forall$ $x > a$, then $\exists$ $a_0 \in (a, b]$ s.t.

$g(t) \neq 0$ $\forall$ $t \in (a, a_0]$ and that

$$\frac{f(t)}{g(t)} \rightarrow l \text{ as } t \rightarrow a^+$$

where $t \in \mathbb{R}$, or $t = \infty$ or $t = -\infty$

Case -I if $g(t) > 0$ $\forall$ $t \in (a, a_0]$

$\int_a^b g(t)\,dt$ is cgt & $l \in \mathbb{R}$,

$\int_{a^+}^b f(t)\,dt$ is cgt

Case-II if $f(t) > 0$ $\forall$ $t \in (a, a_0]$, $\int_{a^+}^b f(t)\,dt$ divergent

then $\int_{a^+}^b g(t)\,dt$ divergent.

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"Convergence of Beta and Gamma function"

Beta function:-

If $m>0$ and $n>0$ then β function of m, n is denoted by $\beta(m,n)$ and it is defined as

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Properties of β function:-

[P1] Beta function is symmetric.

i.e. $\beta(m,n) = \beta(n,m)$

pf Since $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$\beta(m,n) = \int_0^1 (1-x)^{m-1} x^{n-1}\,dx$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}\,dx$$

$\int_0^2 x^2 = \frac{8}{3}$

$\int_0^2 (2-x)^2\,dx =$

$4+x^2-4x$

$\frac{x^3}{3} - \frac{4x^2}{2} + 4x$

$\frac{8}{3} - 8 + 8 = \frac{8}{3}$

[P2] $\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

pf Since $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ —— (1)

Put $x = \frac{1}{1+y}$ $\Rightarrow$ $y = \frac{1}{x} - 1$

$$dx = \frac{-1}{(1+y)^2} dy$$

upper limit $y=0$

L.L. $y = \infty$

$$\therefore \quad \beta(m,n) = \int_{\infty}^{0} \frac{1}{(1+y)^{m-1}} \left(1 - \frac{1}{1+y}\right)^{n-1} \left(-\frac{1}{(1+y)^2}\right) dy$$

$$\Rightarrow \beta(m,n) = \int_0^{\infty} \frac{1}{(1+y)^{m-1}} \cdot \frac{y^{n-1}}{(1+y)^{n-1}} \cdot \frac{1}{(1+y)^2} \cdot dy$$

$$= \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$\Rightarrow \beta(m,n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\left[\int_a^b f(x)\, dx = \int_a^b f(y)\, dy \right]$$

Similarly $\beta(n,m) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{n+m}} dx$

$\Rightarrow \beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx.$

P3 $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Pf

Since $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta \, d\theta$

$\theta = \sin^{-1}\sqrt{x}$

U.L. $\theta = \sin^{-1}(\sqrt{1}) = \frac{\pi}{2}$

L.L. $\theta = \sin^{-1}(0) = 0$

$\therefore \beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (\cos^2\theta)^{n-1} 2\sin\theta\cos\theta \, d\theta$

$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

Gamma Function :-

If $n > 0$, then gamma function of $n > 0$ is denoted by Γn and it is defined as

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

s.t. $\Gamma 0 = \infty$

& $\Gamma -n = \infty$

Properties of Gamma Function :-

P 1 $\quad \Gamma 1 = 1$

pf since $\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$

put $n = 1$.

$$\Gamma 1 = \int_0^\infty e^{-x} \cdot 1 \, dx$$

$$= -\left[e^{-x}\right]_0^\infty = -\left[e^{-\infty} - 1\right]$$

$$= 1$$

P 2 $\quad \Gamma n+1 = n \Gamma n$

pf.

since $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

Put $n = n+1$

$$\Gamma(n+1) = \int_0^\infty \underset{\text{II}}{e^{-x}}\ \underset{\text{I}}{x^n}\, dx$$

$$= \left[x^n(-e^{-x})\right]_0^\infty - \int_0^\infty n x^{n-1}(-e^{-x})\, dx$$

$$\Gamma(n+1) = -\left[\lim_{x\to\infty} \frac{x^n}{e^x} - 0\right] + n\int_0^\infty e^{-x} x^{n-1}\, dx$$

$$= -\lim_{x\to\infty} \frac{x^n}{e^x} + n\,\Gamma(n) \quad \text{--- (1)}$$

Now

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{n x^{n-1}}{e^x}$$

$$\lim_{x\to\infty} \frac{n(n-1)\cdots 3\cdot 2\cdot 1}{e^x}$$

$$= n!\cdot 0$$

$$\Rightarrow \lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

$\Rightarrow$ we get

$$\boxed{\Gamma(n+1) = n\,\Gamma(n)} \quad \#$$

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n)

P3 If n is the integer

then $\Gamma_{n+1} = n!$

pf

Since $\Gamma_{n+1} = n\,\Gamma_n$

$= n(n-1)\,\Gamma_{n-1}$

$= n(n-1)(n-2)\cdots 3\,\Gamma_3$

$= n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$

$= n!$

$\Rightarrow \boxed{\Gamma_{n+1} = n!}$

P4 If z is independent of x then

$$\frac{\Gamma_n}{z^n} = \int_0^\infty e^{-zx}\, x^{n-1}\, dx$$

pf R.H.S

$$= \int_0^\infty e^{-zx}\, x^{n-1}\, dx$$

Put $zx = y \Rightarrow x = \frac{y}{z} \Rightarrow dx = \frac{dy}{z}$

U.L. $y = \infty$, L.L. $y = 0$

$$= \int_0^\infty e^{-y}\cdot \frac{y^{n-1}}{z^{n-1}}\cdot \frac{dy}{z} = \frac{1}{z^n}\int_0^\infty e^{-y}\, y^{n-1}\, dy$$

$$= \frac{1}{z^n}\cdot \Gamma_n = \text{L.H.S.}$$

[Result 1] :-

Show that the integral $\int_0^\infty x^{n-1} e^{-x} dx$ is convergent iff $n > 0$.

Pf

Let $f(x) = x^{n-1} e^{-x} = \dfrac{e^{-x}}{x^{-(n-1)}} = \dfrac{e^{-x}}{x^{1-n}}$

Here f has infinite discontinuity at $x=0$ if $1-n > 0$ or $n < 1$

We need to discuss convergence at 0 and ∞ both

Now $\displaystyle\int_0^\infty x^{n-1} e^{-x} dx = \int_0^1 x^{n-1} e^{-x} dx + \int_1^\infty x^{n-1} e^{-x} dx$

$= I_1 + I_2$

- Now Convergence of I_1 at 0 $(n<1)$

Take $\phi(x) = \dfrac{1}{x^{1-n}}$

$\therefore \dfrac{f(x)}{\phi(x)} = e^{-x}$

then $\displaystyle\lim_{x\to 0} \frac{f(x)}{\phi(x)} = \lim_{x\to 0} e^{-x} = 1$ (finite)

and $\displaystyle\int_0^1 \phi(x)\, dx = \int_0^1 \frac{1}{x^{1-n}} dx$ which is convgt iff $1-n < 1$ or $n > 0$

Now $\lim_{x \to 0} \frac{f(x)}{\phi(x)} = 1$ and $\int_0^1 \phi(x)\, dx$ is convergent for $n>0$

By ~~Practical~~ Comparision test $\int_0^1 f(x)\, dx = I_1$ is also convergent for $n>0$.

* Convergence at ∞ of I_2 :-

Take $\phi(x) = \frac{1}{x^2}$ $\therefore$ $\frac{f(x)}{\phi(x)} = \frac{e^{-x}}{x^{1-n}} \cdot x^2 = \frac{x^{n+1}}{e^x}$

Now $\lim_{x \to \infty} \frac{f(x)}{\phi(x)} = \lim_{x \to \infty} \frac{x^{n+1}}{e^x} = \frac{(n+1)!}{\infty} = 0$

(by L'Hospital R.)

And $\int_1^\infty \frac{1}{x^p}\, dx$ is cgt $\forall$ $P>1$

$\Rightarrow \int_1^\infty \frac{1}{x^2}\, dx$ is convergent.

Now $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$ and $\int_1^\infty \phi(x)\, dx$ is cgt $\forall n$.

By Comparision test $\int_1^\infty f(x)\, dx = I_2$ is cgt $\forall n$

Hence $\int_0^\infty x^{n-1} e^{-x}\, dx$ is cgt iff $n>0$.

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Relation b/w Beta and Gamma function :—

[1] $$\beta(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma m+n} \quad (m>0,\ n>0)$$

Pf

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$$

$$\Gamma n = z^n \int_0^\infty e^{-zx} x^{n-1}\, dx \quad -① \qquad (\text{put } x = zy)$$

$$\Rightarrow \frac{\Gamma n}{z^n} = \int_0^\infty e^{-zx} x^{n-1}\, dx \quad -②$$

multiplying ① by z^{m-1} and e^{-z} and integrate w.r.t. z from $z=0$ to ∞

$$\Gamma n \int_0^\infty e^{-z} z^{m-1}\, dz = \int_0^\infty e^{-z} z^{m-1} \left[z^n \int_0^\infty e^{-zx} x^{n-1}\, dx \right] dz$$

$$\Gamma n\, \Gamma m = \int_0^\infty \int_0^\infty e^{-z(1+x)} z^{m+n-1} x^{n-1}\, dx\, dz$$

by changing the order of integration

$$\Gamma n \cdot \Gamma m = \int_0^\infty x^{n-1} \left[\int_0^\infty e^{-z(1+x)} \cdot z^{m+n-1}\, dz \right] dx$$

$$= \int_0^\infty x^{n-1} \left(\frac{\Gamma m+n}{(1+x)^{m+n}} \right) dx \quad \text{using (2)}$$

$$\Rightarrow \frac{\Gamma n \, \Gamma m}{\Gamma m+n} = \int_0^\infty x^{n-1} \cdot \frac{1}{(1+x)^{m+n}} dx$$

$$\left(\because \beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx \right)$$

(p. 2)

$$\Rightarrow \boxed{\frac{\Gamma n \, \Gamma m}{\Gamma m+n} = \beta(m,n)}$$

#

Prove that $\Gamma \frac{1}{2} = \sqrt{\pi}$

Pf

$$\because \frac{\Gamma m \, \Gamma n}{\Gamma m+n} = \beta(m,n) \quad \text{—(1)}$$

Also by property of beta function

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2m-1}\theta \, d\theta \quad \text{—(2)}$$

By (1) and (2)

$$2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \, d\theta = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}$$

Put $m = 1/2$ and $n = 1/2$

$\Rightarrow$ $2m-1 = 0$ and $2n-1 = 0$

$$\Rightarrow 2\int_0^{\pi/2} d\theta = \frac{\Gamma 1/2 \, \Gamma 1/2}{\Gamma 1}$$

$$= 2 \cdot \pi/2 = \frac{(\Gamma 1/2)^2}{1} \qquad (\because \Gamma 1 = 1)$$

$$\Rightarrow \pi = (\Gamma 1/2)^2$$

$$\Rightarrow \boxed{\Gamma 1/2 = \sqrt{\pi}}$$

Q. $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Sol^n Put $x^2 = z$.

$dx = \frac{1}{2x} dz$

$dx = \frac{1}{2} z^{-1/2} dz.$

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$$\Rightarrow \int_0^{\infty} e^{-x^2} dx = \frac{1}{2}\int_0^{\infty} e^{-z} z^{-1/2} dz$$

$$= \frac{1}{2}\int_0^{\infty} z^{\frac{1}{2}-1} e^{-z} dz.$$

$$= \frac{1}{2}\Gamma\tfrac{1}{2} = \frac{1}{2}\sqrt{\pi}$$

Q Show that $\int_0^1 x^{m-1}(1-x)^{n-1} dx$ Converges iff $m>0$ and $n>0$.

Pf The given integral is proper integral if $m\geq 1, n\geq 1$.

If $m<1$, then 0 is only point of infinite discontinuity and if $n<1$, then 1 is only point of infinite discontinuity.

For $m<1$ and $n<1$, we take $0<\frac{1}{2}<1$

$$\int_0^1 f(x)\,dx = \int_0^{1/2} f(x)\,dx + \int_{1/2}^1 f(x)\,dx$$

when $f(x) = x^{m-1}(1-x)^{n-1}$

Converges at '0' when $m<1$

let $f(x) = \dfrac{(1-x)^{n-1}}{x^{1-m}}$ and $g(x) = \dfrac{1}{x^{1-m}}$

$\therefore \lim_{x \to 0^+} \dfrac{f(x)}{g(x)} = \lim_{x \to 0^+} (1-x)^{n-1} = 1$

Since $\int_0^{1/2} g(x)\,dx = \int_0^{1/2} \dfrac{1}{x^{1-m}}\,dx$ Converges iff $1-m<1$
$\Rightarrow m>0$

So $\int_0^{1/2} x^{m-1} (1-x)^{n-1}\,dx$ Converges at 0 iff $m>0$
(by integral test)

Converges at 1 when $n<1$

let $f(x) = \dfrac{x^{m-1}}{(1-x)^{1-n}}$ and $g(x) = \dfrac{1}{(1-x)^{1-n}}$

$\therefore \lim_{x \to 1^-} \dfrac{f(x)}{g(x)} = 1$

Since $\int_{1/2}^{1} \dfrac{dx}{(1-x)^{1-n}}$ Converge iff $1-n<1$
$\Rightarrow n>0$

$\therefore \int_{1/2}^{1} x^{m-1} (1-x)^{n-1}\,dx$ Converges iff $n>0$

Hence $\int_0^1 x^{m-1} (1-x)^{n-1}\,dx$ Converges iff $m>0$ & $n>0$.

For n, $\Gamma(n+1) = n!$

Solve

Q. $\int_0^1 x^5(1-x)^4\,dx$

Soln

$$\int_0^1 x^5(1-x)^4\,dx = \beta(6,5)$$

$$= \frac{\Gamma 6\,\Gamma 5}{\Gamma 6+5} = \frac{\lfloor 5\times4\times3\times2}{10\times9\times8\times7\times6\times\lfloor 5}$$

$$= \frac{1}{1260}.$$

$\boxed{2}$ $\int_0^\infty x^8 e^{-2x}\,dx$

let $2x=t$, $dx = \frac{dt}{2}$

$$\int_0^\infty x^8 e^{-2x}\,dx = \int_0^\infty \left(\frac{t}{2}\right)^8 e^{-t}\frac{dt}{2}$$

$$= \frac{1}{2^9}\int_0^\infty t^8 e^{-t}\,dt$$

$$= \frac{1}{2^9}\cdot\Gamma 9$$

$$= \frac{\lfloor 8}{2^9} \quad \text{A.}$$

$\boxed{3}$ $\int_0^1 \frac{dx}{1-x^4}$

put $x^4 = z$, $4x^3\,dx = dz \Rightarrow dx = \frac{1}{4}z^{-3/4}\,dz$

$x = z^{1/4}$ (24)

$$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4}\int_0^1 \frac{z^{-3/4}\,dz}{\sqrt{1-z}}$$

$$= \frac{1}{4}\int_0^1 z^{(1/4-1)}(1-z)^{(1/2-1)}\,dz$$

$$= \frac{1}{4}\beta\left(\frac{1}{4},\frac{1}{2}\right)$$

$$= \frac{1}{4}\,\frac{\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{\Gamma\frac{3}{4}}$$

$$= \frac{1}{4}$$

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Exercise - 7.8 (B)

*Ques-17. (l) $\int_1^\infty \frac{x}{\sqrt{x^2-1}}\,dx$.

We can write

$$\int_1^\infty \frac{x}{\sqrt{x^2-1}}\,dx = \int_1^2 \frac{x\,dx}{\sqrt{x^2-1}} + \int_2^\infty \frac{x\,dx}{\sqrt{x^2-1}}$$

& $\int_1^\infty \frac{x\,dx}{\sqrt{x^2-1}}$ converges if $\int_1^2 \frac{x\,dx}{\sqrt{x^2-1}}$ and $\int_2^\infty \frac{x\,dx}{\sqrt{x^2-1}}$ both converges together.

Part I :- Let's define $f(\cdot)$ by

$$f(x) = \frac{x}{\sqrt{x^2-1}} \quad \forall\ x \in]1,2].$$

f is bdd & cts on $[c, 2]$ $\forall$ $1<c<2$ and hence integral thereat.

But, f is not defined on $[1,2]$, thus, f is not integrable on $[1,2]$

$\therefore$ f is an improper integral of type-I which converges if $\lim_{c\to1^+}\int_c^2 \frac{x}{\sqrt{x^2-1}}\,dx$ exists.

So,

for $1<c<2$, consider

$$\int_c^2 \frac{x}{\sqrt{x^2-1}}\,dx$$

$$= \sqrt{x^2-1}\,\Big|_c^2 = \sqrt{3} - \sqrt{c^2-1}$$

& $\lim_{c\to1^+}\int_c^2 \frac{x}{\sqrt{x^2-1}}\,dx = \lim_{c\to1^+}\left[\sqrt{3}-\sqrt{c^2-1}\right]$

$= \sqrt{3}$.

ie $\int_1^2 \frac{x\,dx}{\sqrt{x^2-1}}$ converges to $\sqrt{3}$.

Let $x^2-1 = t^2$

$\cancel{2}x\,dx = \cancel{2}t\,dt$

$\Rightarrow \int \frac{\cancel{t}\,dt}{\cancel{t}} = dt$

$\Rightarrow \sqrt{x^2-1}$

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Part-II). Let's define $f(\cdot)$ by

$$f(x) = \frac{x}{\sqrt{x^2-1}} \quad \forall\ x \in [2,\infty)$$

$\forall\ b>2$, f is cts on on $[2,b]$ and hence integrable there at.

Thus,

f is an improper integral of type-II which converges if $\lim_{b\to\infty} \int_2^b \frac{x\,dx}{\sqrt{x^2-1}}$ exists.

So,

$\forall\ b>2$, consider

$$\int_2^b \frac{x\,dx}{\sqrt{x^2-1}}$$

$$= \sqrt{x^2-1}\,\Big|_2^b$$

$$= \sqrt{b^2-1} - \sqrt{3}.$$

then,

$$\lim_{b\to\infty} \int_2^b \frac{x\,dx}{\sqrt{x^2-1}} = \lim_{b\to\infty} \left[\sqrt{b^2-1} - \sqrt{3}\right] = \infty.$$

i.e. $\int_2^\infty \frac{x\,dx}{\sqrt{x^2-1}}$ diverges to $+\infty$.

Thus,

$$\boxed{\int_1^\infty \frac{x\,dx}{\sqrt{x^2-1}} \text{ diverges to } +\infty.}$$

(m) $\int_1^\infty \frac{x}{(x^2-1)^3}\,dx.$

The given integral can be written as:

$$\int_1^\infty \frac{x\,dx}{(x^2-1)^3} = \int_1^2 \frac{x\,dx}{(x^2-1)^3} + \int_2^\infty \frac{x\,dx}{\sqrt{x^2-1}}$$

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&

$\int_1^\infty \frac{x\,dx}{(x^2-1)^3}$ converges if $\int_1^2 \frac{x\,dx}{(x^2-1)^3}$ & $\int_2^\infty \frac{x\,dx}{(x^2-1)^3}$ both converges together.

Part I :- Let's define $f(\cdot)$ by

$$f(x) = \frac{x}{(x^2-1)^3} \quad \forall\ x \in]1, 2]\,.$$

Here, f is bdd & cts on $[c, 2]$ for $1<c<2$ and hence, f is integrable thereat.

But, since f is not defined at $x=1$, thus, f is not integrable on $[1, 2]$.

$\therefore$ f is an improper integral of type-I which converges if

$\lim_{c\to 1^+} \int_c^2 \frac{x\,dx}{(x^2-1)^3}$ exists.

So, for $1<c<2$, consider

$$\int_c^2 \frac{x\,dx}{(x^2-1)^3}$$

$$= \frac{-1}{2}(x^2-1)^{-1}\Big|_c^2$$

$$= \frac{1}{2(c^2-1)} - \frac{1}{2(3)} = \frac{1}{2(c^2-1)} - \frac{1}{6}.$$

Let $x^2-1 = u$

$2x\,dx = du$

$= \frac{1}{2}\frac{du}{u^3}$

$\int \frac{1}{2} u^{-3} du$

$\frac{1}{2}\frac{u^{-2+1}}{-1}, = \frac{-1}{2u}$

& $\lim_{c\to 1^+} \int_c^2 \frac{x\,dx}{(x^2-1)^3} = \lim_{c\to 1^+}\left[\frac{1}{2(c^2-1)} - \frac{1}{6}\right] = \infty.$

i.e

$\int_1^2 \frac{x\,dx}{(x^2-1)^3}$ diverges to $+\infty$.

Thus, $\boxed{\int_1^\infty \frac{x\,dx}{\sqrt{x^2-1}} \text{ diverges to } +\infty}$

(n) $\int_1^{\infty} \frac{1}{x(\ln x)^2}\,dx$.

We can write

$$\int_1^{\infty} \frac{dx}{x(\ln x)^2} = \int_1^{2} \frac{dx}{x(\ln x)^2} + \int_2^{\infty} \frac{dx}{x(\ln x)^2}$$

So, given Integral converges if $\int_1^2 \frac{dx}{x(\ln x)^2}$ and $\int_2^{\infty} \frac{dx}{x(\ln x)^2}$ both converges together.

Case: 1) $\int_1^2 \frac{dx}{x(\ln x)^2}$

Let's define $f(\cdot)$ by

$$f(x) = \frac{1}{x(\ln x)^2} \quad \forall\, x \in]1,2]$$

for $1<c<2$, f is bdd & cts on $[c,2]$ & hence integrable there;

But,

f is not defined at $x=1$, hence f is not integrable on $[1,2]$

so f is an improper integral of type-I which converges if $\lim_{c \to 1^+} \int_c^2 \frac{dx}{x(\ln x)^2}$ exists.

So, $\forall\ 1<c<2$, consider

$$\int_c^2 \frac{dx}{x(\ln x)^2}$$

$$= \frac{-1}{\log x}\Big|_c^2$$

$$= \frac{1}{\log c} - \frac{1}{\log 2}$$

let $\ln x = u$

$\frac{1}{x}\,dx = du$

$= \int \frac{du}{u^2}$

$= \frac{-1}{u}$

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& $\lim_{c\to1^+} \int_c^2 \frac{dx}{x(\ln x)^2} = \lim_{c\to1^+} \left[\frac{1}{\log c} - \frac{1}{\log 2}\right].$

$= \infty.$

i.e $\int_1^2 \frac{dx}{x(\ln x)^2}$ diverges to $+\infty$.

Thus, $\boxed{\int_1^\infty \frac{dx}{x(\ln x)^2} \text{ diverges to } +\infty.}$

*Ques-27. For each of the following, determine the values of r for which the integral exists or converges and determine the values of integral in those cases:-

(a) $\int_1^\infty \frac{dx}{x^r}$

Let's define $f(\cdot)$ by
$$f(x) = \frac{1}{x^r} \quad \forall\ x \in [1, \infty[$$

$\forall\ b>1$, f is cts. on $[1, b]$ and hence integrable there at.
Thus, f is an improper integral of type-II which converges if $\lim_{b\to\infty} \int_1^b \frac{dx}{x^r}$ exists.

So, $\forall\ b>1$, consider

$$\int_1^b \frac{dx}{x^r}$$
$$= \left.\frac{x^{-r+1}}{-r+1}\right|_1^b$$
$$= \frac{b^{-r+1}}{-r+1} - \frac{1^{-r+1}}{-r+1}$$
$$= \frac{b^{-r+1}}{1-r} - \frac{1}{1-r}$$

& $\lim_{b\to\infty} \int_1^b \frac{dx}{x^r}$

$= \lim_{b\to\infty} \frac{1}{1-r}\left[b^{-r+1} - 1\right]$

$= \frac{1}{1-r}\left[\infty^{-r+1} - 1\right]$

$= \begin{cases} \infty & , \text{ if } r \leq 1 \\ \frac{1}{1-r} & , \text{ if } r > 1 \end{cases}$

i.e $\boxed{\int_1^\infty \frac{dx}{x^r} = \begin{cases} \text{cgs} & \text{for } r > 1 \\ \text{diverges} & \text{for } r \leq 1 \end{cases}}$

(b) $\int_0^1 x^{-r}\,dx$

let's define $f(\cdot)$ by

$$f(x) = x^{-r} \quad \forall\, x \in]0,1].$$

for $0<c<1$, f is bdd & cts. on $[c,1]$ and hence integrable on $[c,1]$.

But, f is not defined at $x=0$; hence, f is not integrable on $[0,1]$

$\therefore$ it is an Improper integral of type I which converges if $\lim_{c\to 0^+} \int_c^1 x^{-r}\,dx$ exists.

So, for $0<c<1$, consider

$\int_c^1 x^{-r}\,dx$

$= \left.\frac{x^{-r+1}}{1-r}\right|_c^1$

$= \frac{1}{1-r} - \frac{c^{-r+1}}{1-r}$

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& $\lim_{c \to 0^+} \int_c^1 x^{-r}\,dx = \lim_{c \to 0^+} \left[\frac{1}{1-r} - \frac{c^{-r+1}}{1-r} \right]$

$$= \begin{cases} \frac{1}{1-r} & \text{if } r<1 \\ \infty & \text{, if } r \geq 1 \end{cases}.$$

thus,

$$\boxed{\int_0^1 x^{-r}\,dx = \begin{cases} \text{cgs} & \text{if } r<1 \\ \text{dvgs} & \text{if } r \geq 1 \end{cases}}$$

(c) $\int_0^\infty x^{-r}\,dx$.

The given integral can be written as

$$\int_0^\infty x^{-r}\,dx = \underbrace{\int_0^1 \frac{dx}{x^r} + \int_1^\infty \frac{dx}{x^r}}$$

& it converges if both these integrals converges together from part (b),

$$\int_0^1 \frac{dx}{x^r} = \begin{cases} \text{cgs} & \text{if } r<1 \\ \text{dvgs} & \text{if } r \geq 1 \end{cases}$$

& from part (a),

$$\int_1^\infty \frac{dx}{x^r} = \begin{cases} \text{cgs} & \text{if } r>1 \\ \text{dvgs} & \text{if } r \leq 1 \end{cases}$$

thus, $\int_0^1 \frac{dx}{x^r} + \int_1^\infty \frac{dx}{x^r} = \begin{cases} \text{dvgs} & \text{for } r \leq 1 \\ \text{dvgs.} & \text{for } r \geq 1 \end{cases}$

thus,

$$\boxed{\int_0^\infty x^{-r}\,dx \text{ diverges for all } x \text{ to } +\infty}$$

Ques-37. Use the comparison test to determine whether the improper integrals converge :

(a) $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x^3+3}}$

for $x > 1$,

$x^3+3 > x^3$

$\Rightarrow \dfrac{1}{x^3+3} < \dfrac{1}{x^3}$

$\Rightarrow 0 < \dfrac{1}{\sqrt{x^3+3}} < \dfrac{1}{\sqrt{x^3}}$

Let $f(x) = \dfrac{1}{\sqrt{x^3+3}}$ & $g(x) = \dfrac{1}{x^{3/2}}$

Now, $g(x)$ converges by p-test. Therefore, by comparison test, $f(x)$ converges.

i.e $\boxed{\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x^3+3}} \text{ converges.}}$

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* Ques-67. $\int_0^\infty \cos x^2 \, dx$

Here, $\lim_{x\to\infty} \cos x^2$ does not exist, since, $\cos x^2$ is an oscillatory func^n, so unique limit doesn't exist.

Now,

$\int_0^\infty \cos x^2 \, dx$ cgs iff $\int_0^1 \cos x^2 \, dx$ & $\int_1^\infty \cos x^2 dx$ cgs together

Proper integral.

Let's define func^n f by

$f(x) = \cos x^2 \quad \forall \ x \in [0, \infty)$

for $b > 0$, f is cts on $[0, b]$ and hence, integrable thereat
$\therefore f$ is an improper integral of type-II which cgs if

$\lim_{b\to\infty} \int_0^b \cos x^2 \, dx$ exists.

Now, $\forall \ b > 0$, consider

$\int_0^b \cos x^2 \, dx$

\# Let $x^2 = u$

$2x \, dx = du$.

$dx = \frac{du}{2x} = \frac{du}{2\sqrt{u}}$.

$\therefore \int \cos x^2 dx = \int \frac{\cos u \cdot du}{2\sqrt{u}}$

$= \frac{1}{2}\left[u^{-1/2} \cdot \sin u - \int \left(\frac{-1}{2}\right) u^{-3/2} \sin u \, du \right]$

$\therefore \frac{\sin x^2}{2x} + \frac{1}{4}\int \frac{\sin x^2}{x^3} \times 2x dx$

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$\Rightarrow \int \cos x^2 dx = \frac{\sin x^2}{2x} + \int \frac{\sin x^2}{2x^2} dx$

$\Rightarrow \int_1^b \cos x^2 dx = \left.\frac{\sin x^2}{2x}\right]_1^b + \underbrace{\int_1^b \frac{\sin x^2}{2x^2} dx}_{\text{Being proper integral, it has exact value}}$

$\Rightarrow \int_1^b \cos x^2 dx$ cgs.

$\Rightarrow \int_0^\infty \cos x^2 dx$ cgs.

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* Exercise 7.3 (b)

* Ques-3). (c) $\int_0^\infty \frac{2x\,dx}{\sqrt{x+1}}$

Here, we will consider two integrals

$$\int_0^1 \frac{2x\,dx}{\sqrt{x+1}} \quad \& \quad \int_1^\infty \frac{2x\,dx}{\sqrt{x+1}}$$

Now,

$\int_0^\infty \frac{2x\,dx}{\sqrt{x+1}}$ cgs iff $\int_0^1 \frac{2x\,dx}{\sqrt{x+1}}$ & $\int_1^\infty \frac{2x\,dx}{\sqrt{x+1}}$ both cgs together.

firstly, take

$$\int_0^1 \frac{2x\,dx}{\sqrt{x+1}}$$

It is a proper integral.

$\therefore$

$\int_0^\infty \frac{2x\,dx}{\sqrt{x+1}}$ cgs iff $\int_1^\infty \frac{2x\,dx}{\sqrt{x+1}}$ cgs.

Consider,

$$\int_1^\infty \frac{2x\,dx}{\sqrt{x+1}}$$

for $x \geq 1$,

$x^3 \geq 1$

& $x^3 \geq x$

$\Rightarrow 2x^3 \geq x+1$

$\Rightarrow \sqrt{2x^3} \geq \sqrt{x+1}$

$\Rightarrow \frac{1}{\sqrt{x+1}} \geq \frac{1}{\sqrt{2x^3}}$

$\Rightarrow \frac{2x}{\sqrt{x+1}} \geq \frac{2x}{\sqrt{2x^3}} = \frac{\sqrt{2}}{x^{1/2}}$

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Thus, we get

$$\underbrace{\frac{\sqrt{2}}{x^{1/2}}}_{\substack{\downarrow \\ \text{dgs by p-test}}} \le \frac{2x}{\sqrt{x+1}}$$

$\therefore \int_1^\infty \frac{2x\,dx}{\sqrt{x+1}}$ dgs

Hence,

$$\int_0^\infty \frac{2x\,dx}{\sqrt{x+1}} \text{ dgs}$$

(c) $\int_{-1}^{\infty} \frac{dx}{x^2+4x+6}$

Let's take two integrals.

$$\int_{-1}^{1} \frac{dx}{x^2+4x+6} \quad \& \quad \int_{1}^{\infty} \frac{dx}{x^2+4x+6}$$

So, $\int_{-1}^{\infty} \frac{dx}{x^2+4x+6}$ cgs iff $\int_{-1}^{1} \frac{dx}{x^2+4x+6}$ & $\int_{1}^{\infty} \frac{dx}{x^2+4x+6}$ both cgs

$\because \int_{-1}^{1} \frac{dx}{x^2+4x+6}$ is a proper integral

$\therefore \int_{-1}^{\infty} \frac{dx}{x^2+4x+6}$ cgs iff $\int_{1}^{\infty} \frac{dx}{x^2+4x+6}$ cgs

Consider,

$$\int_{1}^{\infty} \frac{dx}{x^2+4x+6}$$

It is an improper integral of type-I

So,

for $x \geq 1$,

$$x^2+4x+6 > x^2$$

$$\Rightarrow \frac{1}{x^2+4x+6} < \underbrace{\frac{1}{x^2}}_{\text{cgs by p-test}}$$

$$\therefore \int_1^\infty \frac{dx}{x^2+4x+6} \text{ cgs by comparison-test}$$

$$\Rightarrow \int_{-1}^\infty \frac{dx}{x^2+4x+6} \text{ cgs}$$

OR.

$$x^2+4x+6 = (x+2)^2+2 > (x+2)^2$$

$$\Rightarrow \frac{1}{x^2+4x+6} < \underbrace{\frac{1}{(x+2)^2}}$$

Let $f = \frac{1}{(x+2)^2}$ on $[-1, \infty)$

Prove by II type

(b) $\int_1^\infty \frac{x}{\sqrt{x^3+x}}\, dx$

It's an improper integral of type-II

So,

$\forall\ x \geq 1$,

$$x^3 \geq x$$

$$x^3+x^3 \geq x+x^3$$

$$\Rightarrow 2x^3 \geq x^3+x$$

$$\Rightarrow \sqrt{2x^3} \geq \sqrt{x^3+x}$$

$$\Rightarrow \frac{1}{\sqrt{2}\,x^{3/2}} \leq \frac{1}{\sqrt{x^3+x}}$$

$$\Rightarrow \frac{x}{\sqrt{2}.x^{3/2}} \leq \frac{x}{\sqrt{x^3+x}}$$

$$\Rightarrow \underbrace{\frac{1}{\sqrt{2}.x^{1/2}}}_{\text{dgs by p-test}} \leq \frac{x}{\sqrt{x^3+x}}$$

$$\therefore \int_1^\infty \frac{x\,dx}{\sqrt{x^3+x}} \text{ dgs. By comparison-test.}$$

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(d) $\int_1^\infty \frac{dx}{x\sqrt{x+1}}$

It is an improper integral of type-II

for $x \geq 1$,

$$x+1 > x$$

$$\sqrt{x+1} > \sqrt{x}$$

$$x \cdot \sqrt{x+1} > x^{3/2}$$

$$\Rightarrow \frac{1}{x\sqrt{x+1}} < \underbrace{\frac{1}{x^{3/2}}}_{\text{cgs by p-test}}$$

$\therefore \int_1^\infty \frac{dx}{x\sqrt{x+1}}$ cgs by comparison-test.

(f) $\int_{-\infty}^{\infty} \frac{dx}{x^2+4x+6}$

$\int_{-\infty}^{-3} \frac{dx}{x^2+4x+6}$

$\int_{-3}^{-1} \frac{dx}{x^2+4x+6}$ $\Downarrow$ P. I

$\int_{-1}^{\infty} \frac{dx}{x^2+4x+6}$ $\Downarrow$ cgs by (e) part.

So, take

$$\int_{-\infty}^{-3} \frac{dx}{x^2+4x+6}$$

Here, $x^2+4x+6 = (x+2)^2+2 > (x+2)^2$

$$\Rightarrow \frac{1}{x^2+4x+6} < \frac{1}{(x+2)^2}$$

$\Downarrow$ cgs

$\therefore \int_{-\infty}^{-3} \frac{dx}{x^2+4x+6}$ cgs.

$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2+4x+6}$ cgs.

—x—x—