Discipline Course-I

Semester-II

Paper No: Electricity and Magnetism

Lesson: Chapter 3.2: Ampere's Circuital Law and its Applications

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LEARNING OBJECTIIVES

After going through this chapter, the reader would be able to

- Understand Ampere's circuital law and its importance in calculating the magnetic field.
- Appreciate the limitations of Ampere's circuital law.
- Use Ampere's circuital law for calculating the magnetic field due to a current carrying solenoid and a toroid.
- Do a direct calculation of div $\vec{B}$ and understand the significance of the result.
- Do a direct calculation of curl $\vec{B}$ and understand how this result leads us to Ampere's circuital law.
- Get familiar with the concept of vector potential of magnetic field.
- Understand the significance and importance of the concept of 'vector potential' in calculation of the magnetic field due to a given current distribution.

INTRODUCTION

We start this chapter by stating the Ampere's circuital law. The relevance of this law for calculating magnetic fields, due to symmetric current distributions, is explained by taking the examples of a solenoid and a toroid. The fact that this law can be regarded as an alternative way of the Biot Savart's law, is brought out through an exclusive calculation of the curl of $\vec{B}$. This calculation, incidentally also bring out the inherent limitation of the usual form of this law i.e. its validity holding only for steady currents. We also do a direct calculation of the divergence of $\vec{B}$ and find out that $div\ \vec{B} = 0$.The physical significance of this result i.e. the non-existence of the isolated magnetic poles, is also discussed and explained.

A brief discussion on the concept of 'vector potential',$\vec{A}$, for the magnetic fields, helps us to appreciate the role of this concept in maintaining a 'superficial symmetry' between electric and magnetic fields. The relation between $\vec{B}$ and $\vec{A}$ is brought out and the use of $\vec{A}$, in simplifying the calculations of magnetic fields, for any general current distribution, is briefly discussed.

Ampere's circuital law

This law due to Ampere, provides us with an alternative way of calculating the magnetic field due to a given current distribution. This law, is, in a way, similar to the Gauss's law in electrostatics, which again provides us with an alternative way of calculating the electric field due to a given charge distribution.

Ampere's circuital law states: The line integral of the magnetic field, over a closed path, or loop, equals μ_o times the total current enclosed by that closed loop. We express this law through the mathematical expression:

$$\oint \vec{B} \,.\, d\vec{l} = \mu_o I$$

where I, is the net current enclosed by the loop 'l'.

We shall be discussing the details of a 'proof' of this law (through our calculation of curl of the magnetic field, $\vec{B}$), later on in this unit. We shall than realize that this law can be regarded as an alternative way of stating the Biot Savart's law. We shall also see that the law stated in its above form, is valid only for steady currents. The subsequent generalization of this law, for non-steady currents, by Maxwell, played a crucial role in the development of the electromagnetic theory of light.

A 'Built-in Limitation' of Ampere's circuital law

Ampere's law, as stated above, provides us with an easy, quick and convenient way of calculating the line integral of the magnetic field over a given closed path or loop. We only need to know, or calculate, the (net) current enclosed by that loop. However we need to know the magnetic field itself, rather than just a value of its 'line integral'. Therefore, we need to a special closed path for which one can calculate the magnetic field, rather than just its line integral, through the circuital law. The choice of such a (special) closed path becomes possible only for a limited range of current distributions that have some sort of symmetry or 'idealization' associated with them. Ampere's circuital law, therefore, is a handy tool for calculating the magnetic field only for a (very much) limited range of current distributions. This 'built in' limitation, of this law, restricts its use in practical situation.

Choice of the special closed path or Amperian loop for a given current distribution:

For a closed loop, of length l, say, Ampere's circuital law states that

$$\oint \vec{B} \cdot d\vec{l} = \mu_o I$$

Where I is the net current enclosed by the loop 'l'.

Suppose we are able to choose the loop in such a way that the magnetic field, $\vec{B}$, at any point on the loop, is tangential to the loop, at that point. We can then say that

$$\vec{B} \cdot d\vec{l} = B\, dl \cos 0^o = B\, dl$$

Further, let the loop l be such that $\vec{B}$ has the same magnitude; say (B) at all points on the loop. For the loop satisfying these conditions, we can say

$$\oint \vec{B} \cdot d\vec{l} = \oint B\, dl = B \oint dl = Bl$$

In such a case, we then have, from Ampere's circuital law:

$$\oint \vec{B} \cdot d\vec{l} \,(= \oint B\, dl = B \oint dl = Bl) = \mu_o I$$

$$\therefore B = \frac{\mu_o I}{l}$$

The direction of $\vec{B}$, at any point on the loop, is along the direction of the tangent to the loop at that point. Thus both the magnitude and direction of B become known.

The loop, satisfying the two properties given above, is referred to as the 'Amperian loop' for the given current distribution.

It is possible, however, to think of the 'Amperian loop' in a (slightly) more general way. Let us think of a closed loop l, made up of two parts, say l_1 *and* l_2, such that $\vec{B}$is tangential to the loop at all points on its part l_1 and also has a constant magnitude over all these points. Further, let $\vec{B}$ be normal to the loop at all points on its part l_2. We then have

$$\oint_l \vec{B} \cdot d\vec{l} = \int_{l_1} \vec{B} \cdot d\vec{l} + \int_{l_2} \vec{B} \cdot d\vec{l}$$

$$\oint_l \vec{B} \cdot d\vec{l} = \int_{l_1} \vec{B} \cdot d\vec{l} + 0$$

(because $\vec{B} \cdot d\vec{l} = 0$ *at all the points on the part* l_2 *of the closed loop* l*)*

$$\oint_l \vec{B} \cdot d\vec{l} = B \int_{l_1} dl = Bl_1$$

We would then have, $Bl_1 = \mu_o I$

$$\therefore or\ B = \frac{\mu_o I}{l_1}$$

It is possible, therefore, to again calculate B when one can choose a closed loop for which the parts $l_1\ and\ l_2$ satisfy the (respective) conditions given above. This mixed loop can, therefore, again act as an 'Amperian loop' for its relevant current distribution.

We can, therefore, say that the general Amperian loop, for a given current distribution, needs to have the 'properties' given below:

(1) The direction of $\vec{B}$ should
(i) Either be tangential to the loop at all points
(ii) Or be tangential to the loop over a part of it and normal to the loop over the rest of it.
(2) The magnitude of $\vec{B}$ should have the same constant value over all those points on the loop where the direction of $\vec{B}$ is tangential to the loop.

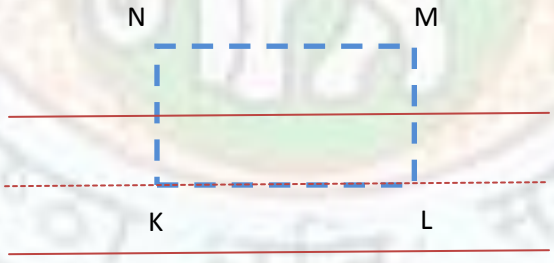
It is easy to realize that it would be very difficult to select, such a loop for any arbitrary/ general current distribution. We can think of Amperian loop only for relatively simple, symmetrical and idealized current distributions. It is the fact that restricts the use of Ampere's circuital law to but a few current distributions.

Applications of Ampere's Circuital law

We would now illustrate the use of Ampere's Circuital law for calculating the magnetic field due to (a) solenoid and (b) a toroid.

(a) Magnetic field due to a (long and thin) solenoid

Consider a long thin solenoid (length of the solenoid>>diameter of the solenoid) having n turns of wire wound uniformly over each unit length of it. For such a solenoid, the magnetic field, when a current flows through its windings, is nearly uniform and parallel to its axis near its center.



To use Ampere's circuital law for calculating the magnetic field, we choose the relevant Amperian loop as a rectangle KLMN. The side KL of this rectangle lies parallel to its axis. The sides LM and KN are perpendicular to the solenoid axis and have such dimensions that make it side NM lie so far away from the solenoid that we can take the magnetic field over this side as practically zero. We then have

$$\oint \vec{B}.d\vec{l} = \oint_{KLMNK} \vec{B}.d\vec{l} = \int_{KL} \vec{B}.d\vec{l} + \int_{LM} \vec{B}.d\vec{l} + \int_{MN} \vec{B}.d\vec{l} + \int_{NK} \vec{B}.d\vec{l}$$

Now $\vec{B} \cdot d\vec{l} = 0$ for the sides LM and NK. This is because $\vec{B}$ and $d\vec{l}$ are perpendicular to each other over either of these two sides.
Further $\vec{B} \cdot d\vec{l} = 0$ over the side MN. This is because this side is assumed to be lie very far away from the solenoid and hence, the magnetic field, due to the solenoid, over this side, can be taken as zero.
Hence,

$$\oint_{KLMNK} \vec{B} \cdot d\vec{l} = \int_{KL} \vec{B} \cdot d\vec{l} = \int_{KL} BdL = B\int_{KL} dL = Ba$$

Where a is the length of the side KL.

Therefore, by Ampere's Circuital law, we have

$$\oint \vec{B} \cdot d\vec{l} = Ba = \mu_o I$$

Where I is the total current enclosed by the rectangular loop KLMNK.

If i is the current through each turn of the solenoid, we have

$$I = (na)i$$

This is because there are (na) turns of the solenoid, each carrying current i, over a length 'a' of the solenoid. Hence

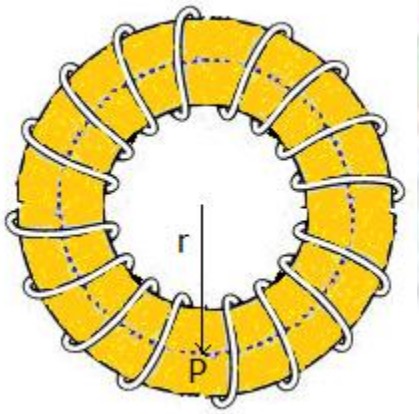
$$Ba = \mu_o (na)i$$

$$\therefore B = \mu_o ni$$

This gives us the (close to the center) axial magnetic field of a long thin solenoid. The direction of this field is along the axis of the solenoid.

(b) <u>Magnetic field due to a toroid:</u>

A toroid (shown in figure below) can be thought of as a long thin solenoid that has been 'closed itself' to form a circle.



The Amperian loop, for an internal point, of a current carrying toroid, can be thought of as a concentric circle passing through this field point, says P. The magnetic field at all points of the loop, is tangential to the loop and has a constant magnitude. Hence

$$\oint \vec{B} \cdot d\vec{l} = \oint B\, dl = B \oint dl = B \cdot 2\pi r$$

By, Ampere's circuital law,

$$\oint \vec{B} \cdot d\vec{l} = \mu_o I$$

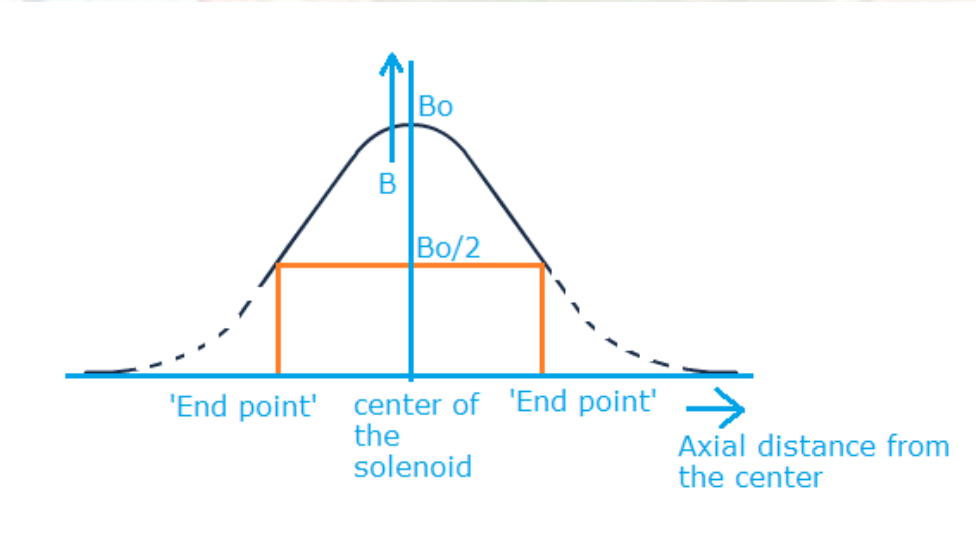
Where I is the net current enclosed by the (Amperian) loop. If i is the current through each turn of the toroid and n is the number of turns per unit 'length' of it, we have

$$I = (i)(n.2\pi r)$$
$$B \cdot 2\pi r = \mu_o.(i)(n.2\pi r)$$
$$or\ B = \mu_o.n\, i$$

This gives us the magnitude of the magnetic field, at an internal point, within the toroid. The direction of the magnetic field is tangential to the loop at all points.

Did You Know

For a solenoid, of a finite length L(for which we cannot assume: length >> diameter) the magnetic field, at axial points, has a nearly constant value over only those axial points that lie close to its center. At the end points of such asolenoid, the magnetic field falls to almost ½ of its value at the close to the center axial points of the solenoid. This point is illustrated by the graoph given below.

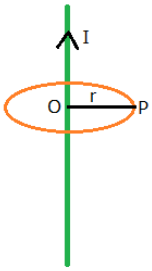


Example: Use the circuital law to calculate the magnetic field due to a long (infinite) straight current carrying wire.

Solution: The symmetry of the setup indicates that we can use a circle, (centered at the foot of the perpendicular from the 'field point' on the wire) as the relevant Amperian loop.

The magnetic field has a constant magnitude at all the points of the circle. By, ampere's law we have

$$\oint \vec{B} \cdot d\vec{l} = \mu_o I$$
$$\therefore \oint B\, dl = \mu_o I$$
$$or\ B \oint dl = \mu_o I$$
$$or\ B.2\,\pi r = \mu_o I$$
$$or\ B = \frac{\mu_o I}{2\pi r}$$



The direction of this magnetic field is along the tangent to the circle at all points.

Did You Know

The Biot Savart's law is written in its usual form as

$$B = \frac{\mu_o}{4\pi} I \int \frac{d\vec{l} \times \vec{r}}{r^3}$$

This form of the law is relevant for linear circuits.

For surface and volume currents, we rewrite the Biot Savart law in the forms.

$$B = \frac{\mu_o}{4\pi} \int \frac{\vec{K} \times \vec{r}}{r^3}$$

And

$$B = \frac{\mu_o}{4\pi} \int \frac{\vec{J} \times \vec{r}}{r^3}$$

Where $\vec{K}$ and $\vec{J}$ denote the surface and volume current densities. In general case of volume currents, $\vec{J}$ here represents the current density vector for the given current distribution.

Example: Consider a long thick cylindrical wire of radius R. Assuming that the current, I, flowing through the wire, is uniformly distributed over the cross section of the wire; calculate the magnetic field at an internal point of the wire.
Solution: We again consider a circle (centered on the foot of the perpendicular from the field point P, on the axis of the wire) as the relevant Amperian loop. The current, enclosed by the loop, is

$$i = I\frac{\pi\, r^2}{\pi\, R^2} = I.\frac{r^2}{R^2}$$

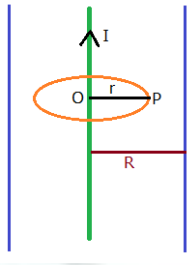
Therefore, by Ampere's circuital law, we have

$\oint \vec{B}\,.d\vec{l} = \mu_o\, i$

$$or\ \ B\oint dl = \mu_o\, I.\frac{r^2}{R^2}$$

$$or\ \ B.2\pi r = \mu_o\, I.\frac{r^2}{R^2}$$

$$or\ \ B = \left(\frac{\mu_o\, I}{2\pi R^2}\right).r$$



Thus the magnetic field, at internal points, for such a wire, is directly proportional to the distance of the field point from the axis of the wire.

Some Results from vector Calculus

We quote below some results from vector calculus that would prove useful during our calculations of the divergence and the curl of the magnetic field.

$$\vec{\nabla}.\left(\vec{A}\times\vec{B}\right) = \vec{B}.\left(\vec{\nabla}\times\vec{A}\right) - \vec{A}.\left(\vec{\nabla}\times\vec{B}\right)$$

$$\vec{\nabla}\times\left(\frac{\vec{P}}{s}\right) = \frac{1}{s}\left(\vec{\nabla}\times\vec{P}\right) + \frac{1}{s^2}\vec{P}\times\left(\vec{\nabla}s\right)$$

$$\vec{\nabla}(r^3) = 3\,\vec{r}.\vec{r}\,\hat{r}$$

$$\vec{\nabla}\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}$$

$$\vec{\nabla}\times\left(\vec{P}\times\vec{Q}\right) = \left(\vec{Q}\,.\vec{\nabla}\right)\vec{P} - \left(\vec{P}\,.\vec{\nabla}\right)\vec{Q} + \vec{P}\left(\vec{\nabla}\,.\vec{Q}\right) - \vec{Q}\left(\vec{\nabla}\,.\vec{P}\right)$$

$$\vec{\nabla}\,.\left(s\vec{P}\right) = s\left(\vec{\nabla}\,.\vec{P}\right) + \vec{P}.\left(\vec{\nabla}s\right)$$

$$or\ \ \vec{P}.\left(\vec{\nabla}s\right) = \vec{\nabla}\,.\left(s\vec{P}\right) - s\left(\vec{\nabla}\,.\vec{P}\right)$$

These results would be used at appropriate places during our calculations of $div\,\vec{B}$ and $curl\,\vec{B}$ that follow hereafter.

Direct Calculation of the divergence of B

We now use the general form of the Biot Savart law, (for a 'volume' current distribution) for doing a direct calculation of the divergence of the magnetic field, i.e. "$div\,\vec{B}$".

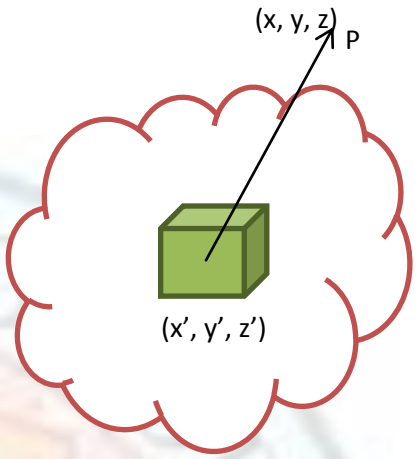
Let there be a good volume current distribution over a volume say τ. The current density vector, at any volume element, $d\tau$, centered around some point, say, (x', y', z'), is denoted by $\vec{J}$. The contribution, $d\vec{B}$, to the magnetic field at some field point, say (x, y, z). due to this volume element is

$$d\vec{B} = \frac{\mu_o}{4\pi}\frac{\left(\vec{J}\,d\tau\right)\times\vec{r}}{r^3}$$

$$d\vec{B} = \frac{\mu_o}{4\pi}\left(\frac{\vec{J}\times\vec{r}}{r^3}\right)d\tau$$

The vector $\vec{r}$, in the expression, is the position vector of the field point (x, y, z) with respect to the center (x', y', z') of the current (volume) element. Hence

$$\vec{r} = (x - x')\hat{\imath} + (y - y')\hat{\jmath} + (z - z')\hat{k}$$



And

$$r = |\vec{r}| = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

The total magnetic field $\vec{B}$, at P, is given by

$$\vec{B} = \int_\tau d\vec{B} = \frac{\mu_o}{4\pi} \int_\tau \left(\frac{\vec{J} \times \vec{r}}{r^3}\right) d\tau$$

Here it is important to remember that

$$\vec{J} = \vec{J}(x', y', z')$$

i.e., the current density vector is a function of the primed coordinates. The integration, indicated above, is to be carried out over all those values of these primed coordinates that are 'contained' in the volume τ.

Now $div\, \vec{B} = \vec{\nabla} . \vec{B}$

$$\vec{\nabla} . \vec{B} = \frac{\mu_o}{4\pi} \vec{\nabla} . \int_\tau \left(\frac{\vec{J} \times \vec{r}}{r^3}\right) d\tau$$

Interchanging, the order of the $\vec{\nabla}$ operation and the integration operation, we get

$$\vec{\nabla} . \vec{B} = \frac{\mu_o}{4\pi} \int_\tau \vec{\nabla} . \left(\frac{\vec{J} \times \vec{r}}{r^3}\right) d\tau$$

Now $\vec{\nabla} . \left(\frac{\vec{J} \times \vec{r}}{r^3}\right) = \frac{\vec{r}}{r^3} (\vec{\nabla} . \vec{J}) - \vec{J} . \left(\vec{\nabla} \times \frac{\vec{r}}{r^3}\right)$

We can say that here, in this case, $(\vec{\nabla} . \vec{J}) = 0$. This is because the operator $\vec{\nabla}$ is being applied here on $\vec{B}$ which is a function of the coordinates (x, y, z) of the field point. The current density vector, $\vec{J}$, however, is a function only of the coordinates (x', y', z') of the points within the volume τ. Clearly, a change in the position of the field point could not affects the value of the current density vector at any point within the volume τ.

Hence,

$$\vec{\nabla} . \left(\frac{\vec{J} \times \vec{r}}{r^3}\right) = 0 - \vec{J} . \left(\vec{\nabla} \times \frac{\vec{r}}{r^3}\right) = -\vec{J} . \left(\vec{\nabla} \times \frac{\vec{r}}{r^3}\right)$$

Now

$$\left(\vec{\nabla} \times \frac{\vec{r}}{r^3}\right) = \frac{1}{r^3} (\vec{\nabla} \times \vec{r}) + \frac{1}{r^6} (\vec{r} \times \vec{\nabla} r^3)$$

It is easy to see that $\left(\vec{\nabla} \times \vec{r}\right) = 0 \; and \; \vec{\nabla}(r^3) = 3\,\vec{r}.\vec{r}\,\hat{r}$
Hence, $\left(\vec{\nabla} \times \frac{\vec{r}}{r^3}\right) = 0 + \frac{1}{r^6}\vec{r} \times 3\,\vec{r}.\vec{r}\,\hat{r} = 0 + 0 = 0$

Hence we see that the integrand, in the integral, for $div\,\vec{B}$, is zero at all points.
Hence

$$div\,\vec{B} = 0$$

Physical significance of the 'zero value' of the $div\,\vec{B}$

We know that the statement

$$div\,\vec{E} = \frac{\rho}{\varepsilon_o}$$

is the differential form of Gauss's law in electrostatics. This leads to its usual

$$\oiint \vec{E}.d\vec{S} = \frac{Q_{enclosed}}{\varepsilon_o}$$

The fact that $div\,\vec{B} = 0$ would, therefore lead us to result

$$\oiint \vec{B}.d\vec{S} = 0$$

i.e., the flux of the magnetic field, through any close surface, is always zero. This implies that there cannot be any net magnetic charge enclosed by the closed surface. We would always have an equal combination of positive and negative magnetic charges i.e. the north and south poles would always exist in pairs.
Thus the mathematical result

$$div\,\vec{B} = 0$$

implies that free or isolated magnetic poles do not exist in nature.

Calculating the curl of $\vec{B}$

This calculation is done by making the same assumptions, and following a similar approach, as done during the calculation of the divergence of $\vec{B}$.
We have

$$curl\,\vec{B} = \vec{\nabla} \times \vec{B}$$

And

$$\vec{B} = \int_\tau d\vec{B} = \frac{\mu_o}{4\pi}\int_\tau \left(\frac{\vec{J}\times\vec{r}}{r^3}\right)d\tau$$

And

$$curl\,\vec{B} = \frac{\mu_o}{4\pi}\,\vec{\nabla} \times \left(\int_\tau \frac{\vec{J}\times\vec{r}}{r^3}\right)d\tau$$

$$or \quad curl\,\vec{B} = \frac{\mu_o}{4\pi}\int_\tau \vec{\nabla} \times \left(\frac{\vec{J}\times\vec{r}}{r^3}\right)d\tau$$

Now $\vec{\nabla} \times \left(\frac{\vec{J}\times\vec{r}}{r^3}\right) = \left(\frac{\vec{r}}{r^3}.\vec{\nabla}\right)\vec{J} - \left(\vec{J}.\vec{\nabla}\right)\frac{\vec{r}}{r^3} + \vec{J}\left(\vec{\nabla}.\frac{\vec{r}}{r^3}\right) - \frac{\vec{r}}{r^3}\left(\vec{\nabla}.\vec{J}\right)$

As explained in the calculation divergence of $\vec{B}$, we would again have

$$\vec{\nabla}.\vec{J} = 0 \; and \; \left(\frac{\vec{r}}{r^3}.\vec{\nabla}\right)\vec{J} = 0$$

$$\therefore \; \vec{\nabla} \times \left(\frac{\vec{J} \times \vec{r}}{r^3}\right) = -\left(\vec{J}.\vec{\nabla}\right)\frac{\vec{r}}{r^3} + \vec{J}\left(\vec{\nabla}.\frac{\vec{r}}{r^3}\right)$$

$$\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

And

$$\vec{\nabla'} = \hat{\imath}\frac{\partial}{\partial x'} + \hat{\jmath}\frac{\partial}{\partial y'} + \hat{k}\frac{\partial}{\partial z'}$$

Since

$$\vec{r} = (x - x')\hat{\imath} + (y - y')\hat{\jmath} + (z - z')\hat{k}$$

It follows that we can put,

$$\vec{\nabla}\left[F\left(\vec{r}\right)\right] = -\vec{\nabla}'\left[F\left(\vec{r}\right)\right]$$

$$\therefore \; -\left(\vec{J}.\vec{\nabla}\right)\frac{\vec{r}}{r^3} = +\left(\vec{J}.\vec{\nabla}'\right)\frac{\vec{r}}{r^3}$$

We first evaluate the result of operating the $\vec{\nabla}'$ operator on the x- component of $\frac{\vec{r}}{r^3}$ which is $\frac{(x-x')}{r^3}$.

From the vector calculus results, we get

$$\left(\vec{J}.\vec{\nabla}'\right)\frac{(x - x')}{r^3} = \vec{\nabla}'.\left(\frac{(x - x')}{r^3}\vec{J}\right) - \frac{(x - x')}{r^3}\left(\vec{\nabla}'.\vec{J}\right)$$

Let us confine our calculations to the case of steady currents. For such currents, the divergence of $\vec{J}$ is zero. We can therefore, put

$$\vec{\nabla}'.\vec{J} = 0$$

for STEADY currents.

$$\therefore \; \left(\vec{J}.\vec{\nabla}'\right)\frac{(x - x')}{r^3} = \vec{\nabla}'.\left(\frac{(x - x')}{r^3}\vec{J}\right)$$

The contribution of this term, to the volume integral, involved in the calculation of $curl\,\vec{B}$, is

$$\int_{\tau} \vec{\nabla}'.\left(\frac{(x - x')}{r^3}\vec{J}\right) d\tau$$

This volume integral can be converted into surface integral

$$\iint_{S} \frac{(x - x')}{r^3}\vec{J}.d\vec{S}$$

by using the divergence theorem.

Suppose we replace the volume τ, enclosing the current distribution, by a (very much) larger volume under the condition that the additional volume does not enclose any currents. The volume integral, over the two volumes would then be the same. However, there being no surface currents on the surface of the larger volume, the surface integral (equivalent to the volume integral) would tend to zero. This implies that the volume integral

$$\int_\tau \vec{\nabla}'.\left(\frac{(x-x')}{r^3}\vec{J}\right) d\tau$$

must be zero. The same result would hold for the other two components of $\frac{\vec{r}}{r^3}$. Hence

$$\int_\tau \vec{\nabla}'.\left(\frac{\vec{r}}{r^3}\vec{J}\right) d\tau = 0$$

We are thus left with only one term in the volume integral involved in the calculation of $\vec{B}$. we can therefore, put

$$curl\,\vec{B} = \frac{\mu_o}{4\pi}\int_\tau \vec{J}.\vec{\nabla}\left(\frac{\vec{r}}{r^3}\right) d\tau$$

The x component, of the integral on the right, can be written as

$$\frac{\mu_o}{4\pi}\int_\tau J_x.\vec{\nabla}\left(\frac{\vec{r}}{r^3}\right) d\tau$$

This can be evaluated by using an analogy from electrostatics. In electrostatics, we have

$$div\,\vec{E} = \frac{\rho}{\varepsilon_o}$$

For a volume charge distribution, confined to a volume τ, and specified by a charge density function $\rho = \rho(x',y',z')$, we can write

$$\vec{E} = \frac{1}{4\pi\varepsilon_o}\int_\tau \rho(x',y',z')\left(\frac{\vec{r}}{r^3}\right)d\tau$$

$$\vec{\nabla}.\vec{E} = \frac{1}{4\pi}\int_\tau \frac{\rho}{\varepsilon_o}\left(\vec{\nabla}.\frac{\vec{r}}{r^3}\right)d\tau$$

Here we have again interchanged the order of carrying out the operations corresponding to the $\vec{\nabla}$ operator and the integration operation. Now since

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_o}$$

Comparing this result within the above integral, involving J_x, we can write

$$\frac{\mu_o}{4\pi}\int_\tau J_x \,.\, \vec{\nabla}\left(\frac{\vec{r}}{r^3}\right) d\tau$$

as equal to $\mu_o J_x$. Similar results would hold for the other two components, namely $J_y \; and \; J_z$.

It follows that

$$\frac{\mu_o}{4\pi}\int_\tau \vec{J} \,.\, \vec{\nabla}\left(\frac{\vec{r}}{r^3}\right) d\tau = \mu_o J_x \,\hat{\imath} + \mu_o \, J_y \hat{\jmath} + \, \mu_o \, J_z \hat{k} = \, \mu_o \vec{J}$$

But the integral equals $curl\, \vec{B}$ for the case of steady currents. We can, therefore, say that corresponding to steady currents, we would have

$$curl\, \vec{B} = \mu_o \vec{J}$$

Thus $curl\, \vec{B}$ is non zero even for steady current distributions. The magnetic field thus has a rotational nature even for steady currents. One can also say that the magnetic field, even due to steady currents, is non conservative in nature.

A 'Proof' of Ampere's Circuital Theorem

The above calculation, of $curl\, \vec{B}$, immediately leads us to Ampere's circuital theorem. We notice that

$$curl\, \vec{B} = \mu_o \vec{J}$$

It follows that, for a closed surface S

$$\oiint curl\, \vec{B} \,.\, d\vec{S} = \mu_o \iint \vec{J} \,.\, d\vec{S}$$

By stoke's theorem,

$$\oiint curl\, \vec{B} \,.\, d\vec{S} = \oint \vec{B} \,.\, d\vec{l}$$

Also $\iint \vec{J} \,.\, d\vec{S} = I$, where I is the total current enclosed by the surface S. We thus get

$$\oint \vec{B} \,.\, d\vec{l} = \, \mu_o \, I$$

In other words: the line integral of the magnetic field, over a closed loop l, equals $\mu_o$times the total current (I) enclosed by thin loop. This is nothing but Ampere's Circuital theorem.

Two important points need to be kept in mind here:

(i) The result, $curl\, \vec{B} = \mu_o \vec{J}$, which leads us to Ampere's Circuital law, has been obtained by using the value of $\vec{B}$ specified by the Biot Savart Law. It is for this reason that we refer to Ampere's Circuital law as just an alternative way of stating the Biot Savart law.

(ii) The result for $curl\, \vec{B}$ has been obtained by assuming that the currents involved are steady currents. It follows that the results for $curl\, \vec{B}$, and therefore, the Ampere's Circuital law (that follows from this result) are valid, in their given forms, only for steady currents.

Did You Know
The limitation (valid only for steady currents) of the conventional form of Ampere's circuital law was pointed out by Maxwell. He introduced the concept of displacement currents (currents associated with time varying electric fields) to generalize Ampere's Circuital law even for non steady currents. It was this generalization (of Ampere's circuital law) that played a crucial and central role in the development of Maxwell's electromagnetic theory of light.

Vector Potential

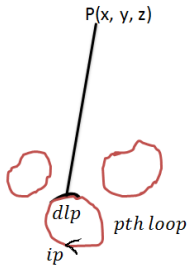
The concept, of potential of a field, plays a useful role in studying and calculating different types of fields. For the electrostatic (and gravitational) field, we can define a scalar function whose 'rate of change' can be related to the components of the field, and therefore, to the field itself. It is possible to define such a scalar function for these fields because these fields are conservative in nature.

The magnetic field, as we have seen above, is non conservative I nature. This follows from the non-zero value of the curl of the field. It is, therefore, not feasible to define a scalar function for this field in the way it can be done for the electrostatic (or gravitational) field. However, to maintain a semblance of symmetry between the two fields, and to have a simpler approach for calculating the magnetic field, it was found to be reasonable to introduce and define, a new vector function for the magnetic field. This vector function was given the name vector potential. It is denoted by the symbol, $\vec{A}$, and as we shall see, the mathematics involved in the calculations of $\vec{A}$ is simpler than that involved in a direct calculation of the magnetic field. Once the vector potential, $\vec{A}$, corresponding to a given current distribution, has been calculated out, we can use the known expression for $\vec{A}$ to calculate $\vec{B}$ through the relation

$$\vec{B} = Curl\ \vec{A}$$

Let us now see how the vector potential $\vec{A}$, for a given (general) current distribution, can be defined. Let there be n current loops, producing the magnetic field, in a given region of space. We can then write, as per Biot – Savart law, the total magnetic field, at ny field point $P(x, y, z)$, as

$$\vec{B} = \frac{\mu_o}{4\,\pi} \sum_{p=1}^{n} i_p \int \frac{d\vec{l_p} \times \vec{r_p}}{{r_p}^3}$$



Now $d\vec{l_p}$ can be written as

$$\vec{l_p} = \hat{\imath}\, dx' + \hat{\jmath}\, dy' + \hat{k}\, dz'$$

where (x', y', z') are the coordinates of the point about which the current element $d\vec{l_p}$ is located. It follows that

$$\overrightarrow{r_p} = (x - x')\hat{\imath} + (y - y')\hat{\jmath} + (z - z')\hat{k}$$

Using the result

$$\vec{\nabla}\left(\frac{1}{r_p}\right) = \frac{-\overrightarrow{r_p}}{{r_p}^3}$$

We can write

$$d\overrightarrow{l_p} \times \frac{\overrightarrow{r_p}}{{r_p}^3} = \vec{\nabla}\left(\frac{1}{r_p}\right) \times d\overrightarrow{l_p}$$

We again need to remind ourselves that the operator $\vec{\nabla}\left(= \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$ does not act on the primed coordinates defying the current elements in any loop. This fact makes it possible for us to write

$$\vec{\nabla}\left(\frac{1}{r_p}\right) \times d\overrightarrow{l_p} = \vec{\nabla} \times \left(\frac{d\overrightarrow{l_p}}{r_p}\right) = curl\left(\frac{d\overrightarrow{l_p}}{r_p}\right)$$

It follows that we can now write

$$\vec{B} = \frac{\mu_o}{4\pi}\sum_{p=1}^{n} i_p \int curl\left(\frac{d\overrightarrow{l_p}}{r_p}\right)$$

Interchanging the order of applying the integration and the curl operators, we can now write

$$\vec{B} = \frac{\mu_o}{4\pi}\sum_{p=1}^{n} i_p \, curl \int \left(\frac{d\overrightarrow{l_p}}{r_p}\right)$$

$$\vec{B} = curl\left[\sum_{p=1}^{n}\frac{\mu_o i_p}{4\pi}\int\left(\frac{d\overrightarrow{l_p}}{r_p}\right)\right]$$

Denoting the whole term, on which, on which the curl operator operates, by $\vec{A}$, we can write

$$\vec{B} = curl\,\vec{A}$$

Where

$$\vec{A} = \sum_{p=1}^{n}\frac{\mu_o i_p}{4\pi}\int\left(\frac{d\overrightarrow{l_p}}{r_p}\right)$$

The vector function $\vec{A}$, defined in this way, is known as the 'vector potential' for the given magnetic field. It is easy to see that the integrals involved, in the calculation of $\vec{A}$, are simpler than those involved in a direct calculation of the magnetic field $\vec{B}$. Once $\vec{A}$ has been worked out, one can get $\vec{B}$ through the relation

$$\vec{B} = curl\,\vec{A}$$

The concept of vector potential thus provides an apparently simpler way of calculating the magnetic field due to a general current distribution.

<u>Need for Adding another condition on $\vec{A}$</u>

Let $\vec{A}$ be the vector potential associated with the magnetic field, $\vec{B}$, due to a given current distribution. We then have

$$\vec{B} = curl\ \vec{A}$$

Let there be a new function, say $\vec{A}'$, which is related to $\vec{A}$ through the relation

$$\vec{A}' = \vec{A} + grad(\lambda)$$

Where $(\lambda)$is any scalar function. We then have

$$curl\ \vec{A}' = curl\ \vec{A} + curl\ grad(\lambda) = curl\ \vec{A} = \vec{B}$$

$$\because\ curl\ grad(\lambda) = 0$$

The function $\vec{A}'$ can, therefore, be also viewed as the 'vector potential' for the given magnetic field, $\vec{B}$. It follows that if we only compose the condition

$$curl\ \vec{A} = \vec{B}$$

On the vector potential, the vector potential, defined in this way would be arbitrary to the extent of addition of the gradient of any scalar function to it.

We thus realize that we need to have additional condition on $\vec{A}$ so that the vector potential, for a given magnetic field, can be defined in a unique, non – arbitrary, way. This additional condition has been put as

$$div\ \vec{A} = 0$$

Thus we say that the vector potential, $\vec{A}$, for a given magnetic field $\vec{B}$, must first be calculated from its defining integral

$$\vec{A} = \sum_{p=1}^{n} \frac{\mu_o i_p}{4\,\pi} \int \left(\frac{d\overrightarrow{l_p}}{r_p}\right)$$

And then, if need be, the gradient of a suitable function, λ, be added to it so that $div\ \vec{A}$ becomes zero. The vector function, $\vec{A}$, for which $div\ \vec{A} = 0$, would then be used for calculating the magnetic field $\vec{B}$ through the relation

$$\vec{B} = curl\ \vec{A}$$

Did You Know ?

In terms of the vector potential $\vec{A}$ (that satisfies both the conditions $div\ \vec{A} = 0$ and $\vec{B} = curl\ \vec{A}$), Ampere's circuital law

$$curl\ \vec{A} = \mu_o \vec{J}$$

takes the form

$$curl\ curl\ \vec{A} = \mu_o \vec{J}$$

$$\vec{\nabla}\left(div\ \vec{A}\right) - \vec{\nabla}^2\ \vec{A} = \mu_o \vec{J}$$

$$-\vec{\nabla}^2 \vec{A} = \mu_o \vec{J}$$

$$or\ \vec{\nabla}^2 \vec{A} = -\mu_o \vec{J}$$

This equation has a form similar to the Poisson's equation ($\vec{\nabla}^2 V = -\frac{\rho}{\varepsilon_o}$ in electrostatics.

Summary

1. Ampere's circuital law states: The line integral of the magnetic field, over a closed path, or loop, equals μ_o times the total current enclosed by that closed loop. We express this law through the mathematical expression:

$\oint \vec{B} . d\vec{l} = \mu_o I$

where I, is the net current enclosed by the loop 'l'.

2. We need a special closed path, called the Amperian loop, for using the circuital law to calculate the magnetic field due to a given current distribution.
3. The Amperian loop needs to satisfy the following properties:
   (i) The direction of $\vec{B}$ should (a) either be tangential to the loop at all points (b) Or be tangential to the loop over a part of it and normal to the loop over the rest of it.
   (ii) The magnitude of $\vec{B}$ should have the same constant value over all those points on the loop where the direction of $\vec{B}$ is tangential to the loop.
4. We can use of Ampere's Circuital law for calculating the magnetic field due to (a) solenoid and (b) a toroid.
5. We can now use the generalized form of the Biot Savart law, to do a direct calculation of "$div\ \vec{B}$" and "$curl\ \vec{B}$".
6. It turns out that $div\ \vec{B} = 0$. This implies that free or isolated magnetic poles do not exist in nature.
7. It turns out that $curl\ \vec{B} = \mu_o \vec{J}$. This non zero value of $curl\ \vec{B}$ implies that the magnetic field is non conservative in nature.
8. The results:

$$curl\ \vec{B} = \mu_o \vec{J}$$
$$and\ \oint \vec{B} . d\vec{l} = \mu_o I$$

   Are equivalent to each other. The first of these is also referred to as the differential form of Ampere's circuital law.

9. The Ampere's Circuital law is valid, only for steady currents. It was generalized later, even for non-steady currents, by Maxwell through his concept of displacement current.
10. Because of the non-conservative nature of the magnetic field, we cannot associate a scalar potential function with this field.
11. We still introduce a 'vector potential' with magnetic field.
12. The vector potential $\vec{A}$, associated with a given current distribution, is defined through the integral $\vec{A} = \frac{\mu_o i_p}{4\pi} \int \left(\frac{d\vec{l_p}}{r_p}\right)$, for a given current loop.

13. The vector potential $\vec{A}$, is related to the magnetic field,$\vec{B}$, as

$$\vec{B} = curl\ \vec{A}$$

14. To remove the arbitrariness, associated with the vector potential, we need to impose an additional condition on the acceptable form of this function. This condition is

$$div\ \vec{A} = 0$$

15. The vector potential, $\vec{A}$, satisfies the equation

$$\vec{\nabla}^2\ \vec{A} = -\mu_o\ \vec{J}$$

which is similar to the Poisson's equation in electrostatics.

Questions

1. Fill in the blanks:
   (i) Ampere's Circuital law relates the _____ _____ of the magnetic field with the 'net enclosed current'.
   (ii) A _____ can be viewed as a solenoid that has been bent and closed itself.
   (iii) The product of μ_o, with the current density vector, at afield point equals the _____ of the magnetic field at that point.
   (iv) The non-existence of free magnetic poles is expressed through the equation _____.
   (v) The form of the circuital law, as given by Ampere, is valid____ for ____ currents.

   Answers

(i) Line integral
(ii) toroid
(iii) $curl$
(iv) $div\ \vec{B} = 0$
(v) only; steady

True of False

State whether the following statements are 'true' or 'False'.

(i) We can select any closed loop for calculating the magnetic field using the circuital law.
(ii) The circuital law helps us to calculate the magnetic field only for a limited number of idealized or symmetric current distributions.
(iii) The non-zero value of $curl\ \vec{B}$ implies that the magnetic field is conservative in nature.
(iv) The zero value of $div\ \vec{B}$ implies that isolated, free magnetic poles do not exist.
(v) The potential, defined for a magnetic field, is a vector quantity.

Answers

(i) False (we need to select an appropriate Amperian loop for a given current distribution)
(ii) True (This is correct statement)
(iii) False(It implies that the magnetic field is non-conservative in nature)
(iv) True (This is correct statement)
(v) True (This is correct statement)

Multiple Choice Questions

Select the best alternative in each of the following:

(i) Ampere's circuital law, valid for steady currents, is expressed
(a) Only through the equation: $\oint \vec{B} \cdot d\vec{l} = \mu_o I$
(b) Only through the equation: $curl\ \vec{B} = \mu_o \vec{J}$
(c) Either through the equation: $\oint \vec{B} \cdot d\vec{l} = \mu_o I$ or through the equation $curl\ \vec{B} = \mu_o \vec{J}$
(d) neither through the equation: $\oint \vec{B} \cdot d\vec{l} = \mu_o I$ nor through the equation $curl\ \vec{B} = \mu_o \vec{J}$

(ii) When using the circuital law, for calculation of magnetic fields, we need to select an appropriate:
(a) Loop for given current distribution.
(b) Surface for the given current distribution.
(c) Closed loop (enclosing a surface) for the given current distribution.
(d) Closed surface (enclosing a volume) for the given current distribution.

(iii) The relations valid for the magnetic field due to a general current distributions, are
(a) $\vec{\nabla} . \vec{B} = 0$; $\vec{\nabla} \times \vec{B} \neq 0$.
(b) $\vec{\nabla} . \vec{B} \neq 0$; $\vec{\nabla} \times \vec{B} = 0$.
(c) $\vec{\nabla} . \vec{B} \neq 0$; $\vec{\nabla} \times \vec{B} \neq 0$.
(d) $\vec{\nabla} . \vec{B} = 0$; $\vec{\nabla} \times \vec{B} = 0$.

(iv) For the magnetic field, associated with a current distribution, we can define, in a unique way,
(a) Only a scalar potential.
(b) Only a vector potential.
(c) Both a scalar potential as well as a vector potential.
(d) neither a scalar potential nor a vector potential. .

(v) The appropriate vector potential , $\vec{A}$, corresponding to given current distribution, has to satisfy

(a) Only the condition $\vec{\nabla} \times \vec{A} = \vec{B}$
(b) Only the condition $\vec{\nabla} . \vec{A} = 0$
(c) Both the conditions: $\vec{\nabla} \times \vec{A} = \vec{B}\ and\ \vec{\nabla} . \vec{A} = 0$
(d) Neither the conditions: $\vec{\nabla} \times \vec{A} = \vec{B}\ nor\ \vec{\nabla} . \vec{A} = 0$

Answers

(i) (c)
Justification/Feedback for the correct answer:
The Ampere's circuital law is expressed through the equation

$$\oint \vec{B} \cdot d\vec{l} = \mu_o I$$

This equation can be put in its equivalent differential form as

$$curl\ \vec{B} = \mu_o \vec{J}$$

Hence both the equations correspond to Ampere's circuital law. Hence choice (c) is correct and choices (a), (b) and (d) are incorrect.

(ii) (c)

Justification/Feedback for the correct answer:

We can use Ampere's circuital law only if we can select a close (Amperian) loop for the given current distribution. This loop has to satisfy certain characteristic properties. We can use the circuital law only when we are able to select such an appropriate closed loop for the given current distribution. Hence choice (c) is correct and choices (a), (b) and (d) are incorrect.

(iii) (a)

Justification/Feedback for the correct answer:

The magnetic field due to any general current distribution, is known to be divergence less i.e. $\vec{\nabla}.\vec{B} = 0$. Also, because of its non-conservative nature, it, in general, has a finite (non zero) value for its curl. Thus $\vec{\nabla} \times \vec{B} \neq 0$. (in fact, $\vec{\nabla} \times \vec{B} = \mu_o \vec{J}$). Thus,

$\vec{\nabla}.\vec{B} = 0; \vec{\nabla} \times \vec{B} \neq 0.$

Hence choice (a) is correct, and choices (b), (c), and (d) are incorrect.

(iv) (b)

Justification/Feedback for the correct answer:

The magnetic field, due to a current distribution, is non conservative in nature. We cannot, therefore, define a unique vector potential, for this field. It is, however, possible to define a unique vector potential, for this field. This unique vector potential has to satisfy two conditions:

$$\vec{\nabla} \times \vec{A} = \vec{B} \; and \; \vec{\nabla}.\vec{A} = 0$$

And we can define a unique vector function that meets both these requirements. Hence choice (b) is correct, and choices (a), (c) and (d) are incorrect.

(v) (c)

Justification/Feedback for the correct answer:

The vector potential $\vec{A}$ is defined so that one can calculate the magnetic field from it through the condition Both the conditions: $\vec{\nabla} \times \vec{A} = \vec{B}$. However, this condition alone is not sufficient to define $\vec{A}$ unambiguously. We can add the gradient of any scalar to the value of $\vec{A}$ and still have$\vec{\nabla} \times \overrightarrow{A'} = \vec{B}$ $(\vec{A}' = \vec{A} + grad(\lambda))$. This is because curl grad = 0. We, therefore, need another condition on $\vec{A}$ so that we may define it in a unique, unambiguous manner. The additional condition imposed is $\vec{\nabla}.\vec{A} = 0$. Hence choice (c) is correct, and choices (a), (b), and (d) are incorrect.

Short note type:

Give brief answers to the following questions:

(a) State Ampere's circuital law.
(b) What are the conditions that 'Amperian loop' needs to satisfy?
(c) Why can't we use the circuital law for calculating the magnetic field due to any arbitrary current distribution?

(d) State the Physical significance of the result $\vec{\nabla}.\vec{B} = 0$ and give a brief justification for the same.
(e) Why is it necessary to impose an additional condition on $\vec{A}$ so that it can be defined in a non-arbitrary manner?

Essay Type

(a) Use Ampere's circuital law, to calculate the axial magnetic field due to a long, thin, current carrying solenoid.
(b) Define the toroid. Obtain the expression for the (internal) magnetic field due to current carrying circular toroid.
(c) Do appropriate calculations to show that $\vec{\nabla}.\vec{B} = 0$.
(d) State the value of $curl\ \vec{B}$ and use it to obtain the usual form of the circuital law.
(e) Why, and how, do we define the vector potential for a magnetic field?