Ex 6.1

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(15) (a) If $x = 0 = 0y$ or $y = 0 = 0x$

Then

$\langle x, y\rangle = \langle 0, y\rangle = 0$

or $\langle x, y\rangle = \langle x, 0\rangle = 0$

$\therefore |\langle x, y\rangle| = 0$

as $\|x\| = 0$ or $\|y\| = 0$

So, $\|x\|\|y\| = 0$

Thus $|\langle x, y\rangle| = \|x\|\|y\|$

If $x = ay$ or $y = bx$ for some scalars a and b

Then $\langle x, y\rangle = \langle ay, y\rangle = a\langle y, y\rangle = a\|y\|^2$

or

$\langle x, y\rangle = \langle x, bx\rangle = \bar{b}\langle x, x\rangle = \bar{b}\|x\|^2$

$\Rightarrow |\langle x, y\rangle| = |a|\|y\|^2 = |a|\|y\|^2$

or $|\langle x, y\rangle| = |\bar{b}|\|x\|^2 = |b|\|x\|^2$

as $\|x\|\|y\| = \|ay\|\|y\|$ or $\|x\|\|bx\|$

$= |a|\|y\|\cdot\|y\|$ or $\|x\||b|\|x\|$

$\Rightarrow |a|\|y\|^2$ or $|b|\|x\|^2$

$\therefore |\langle x, y\rangle| = \|x\|\|y\|$

— x —

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(i) Result

If $\|x+y\| = \|x\| + \|y\|$

then, one of x or y is a multiple of the other

Now,

$\|x+y\|^2 = (\|x\| + \|y\|)^2$

$\Rightarrow \|x\|^2 + 2\,Re\langle x,y\rangle + \|y\|^2 = \|x\|^2 + \|y\|^2 + 2\|x\|\,\|y\|$

$\Rightarrow Re\langle x,y\rangle = \|x\|\|y\|$

$\Rightarrow \|x\|\,\|y\| \le |\langle x,y\rangle|$ $(\because Re\, z \le |z|)$

But $|\langle x,y\rangle| \le \|x\|\,\|y\|$ (By Cauchy Schwarz Inequality).

So, $|\langle x,y\rangle| = \|x\|\|y\|$

$\therefore$ x or y is a multiple of the other

Conversely

The converse is not necessarily true

let $x = (1, 0, -1, 0)$

$y = (-2, 0, 2, 0) \in \mathbb{R}^4$

Then $y = -2x$

$x+y = (-1, 0, 1, 0)$

$\therefore \|x+y\| = \sqrt{(-1)^2 + 0^2 + 1^2 + 0^2} = \sqrt{2}$

$\|x\| = \sqrt{2}$

$\|y\| = 2\sqrt{2}$

$\therefore \|x+y\| \neq \|x\| + \|y\|$

—— x ——

(5) For n vectors

If $\|x_1 + x_2 + \cdots + x_n\|$

$$= \|x_1\| + \|x_2\| + \cdots + \|x_n\|$$

~~then $x_i = x_j$ is a multiple of the other~~

$$\|x_1 + x_2 + \cdots + x_n\|^2 = \|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2 + 2Re\langle x_1, x_2\rangle + 2Re\langle x_1, x_3\rangle + \cdots + 2Re\langle x_1, x_n\rangle + 2Re\langle x_2, x_3\rangle + 2Re\langle x_2, x_4\rangle + \cdots + 2Re\langle x_2, x_n\rangle + \cdots + 2Re\langle x_{n-1}, x_n\rangle$$

and $(\|x_1\| + \|x_2\| + \cdots + \|x_n\|)^2$

$$= \|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2 + 2\|x_1\|\|x_2\| + 2\|x_1\|\|x_3\| + \cdots + 2\|x_1\|\|x_n\| + 2\|x_2\|\|x_3\| + \cdots + 2\|x_2\|\|x_n\| + \cdots + 2\|x_{n-1}\|\|x_n\|$$

$$\Rightarrow \sum_{\substack{i,j=1 \\ i\neq j}}^{n} Re\langle x_i, x_j\rangle = \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \|x_i\|\|x_j\|$$

$$\Rightarrow \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \|x_i\|\|x_j\| \leq \sum_{\substack{i,j=1 \\ i\neq j}}^{n} |\langle x_i, x_j\rangle|$$

But $|\langle x_i, x_j\rangle| \leq \|x_i\|.\|x_j\|$ for $i \geq 1$

(by Cauchy Schwarz inequality)

$$\therefore \sum_{i,j=1}^{n} |\langle x_i, x_j\rangle| \leq \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \|x_i\|\|x_j\|$$

So,
$$\sum_{\substack{i,j=1 \\ i\neq j}}^{n} |\langle u_i, x_j\rangle| = \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \|u_i\| \, \|u_j\|$$

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~~Definition~~

~~Let~~

Remark

(1) If x and y are vectors in R^3 or R^2 then $\langle x,y\rangle = \|x\|\,\|y\| \cos\theta$, where $0 \le \theta \le \pi$) is the angle between x and y

$$\Rightarrow |\cos\theta| = \left|\frac{\langle x,y\rangle}{\|x\|\,\|y\|}\right| = \frac{|\langle x,y\rangle|}{\|x\|\,\|y\|}$$

Since $|\cos\theta| \le 1$

$$\Rightarrow \frac{|\langle x,y\rangle|}{\|x\|\,\|y\|} \le 1$$

$$\Rightarrow |\langle x,y\rangle| \le \|x\|\,\|y\|$$

(2) If x and y are non-zero vectors then x and y are perpendicular iff $\cos\theta = 0$

i.e. iff $\langle x,y\rangle = 0$.

Definition

Let V be an inner product space. vectors x and y in V are said to be orthogonal (perpendicular) if $\langle x,y\rangle = 0$.

~~A Subset~~

Definition

A Subset S of an inner product Space V is said to be orthogonal if any two i.e. those vectors in S are orthogonal vectors

i.e. if $u_i = u_j$ are any two vectors in S then $\langle u_i, u_j \rangle = 0$

Definition

A vector x in non-inner product space V is called a unit vector $\|u\| = 1$

Definition

A subset S of an inner product space V is called an orthogonal set if S is an orthogonal set of unit vectors.

i.e. If $u_i \neq u_j$ are any two vectors in S, then $\langle u_i, u_j \rangle = 0$

as $\|u_i\| = 1$ for each i.

Note:-

① If $S = \{u_1, u_2, \ldots\}$ is an orthonormal set then $\langle u_i, u_j \rangle = 0$ for $i \neq j$

as $\langle u_i, u_i \rangle = 1$ for each i

i.e. $\langle u_i, u_j \rangle = \delta_{ij}$

② If we multiply ~~any nonzero vector~~ the vectors by nonzero scalars ~~than~~ their orthogonality is not affected

$\langle a u_i, a v_j \rangle$

$= a\bar{a} \langle u_i, u_j \rangle$

$= 0$ for $i \neq j$

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(3) If x is a non-zero vector then $\|x\| \neq 0$

as $\left\| \frac{x}{\|x\|} \right\| = \left| \frac{1}{\|x\|} \right| \|x\| = 1$

So, $\frac{x}{\|x\|}$ is a unit vector

This process of multiplying a non-zero vector by a reciprocal of its length called normalizing.

Example-8

In $\mathbb{R}^3$ the set $\{(1,1,0), (1,-1,1), (-1,1,2)\}$ is an orthogonal set of nonzero vectors

Let $v_1 = (1,1,0)$, $v_2 = (1,-1,1)$, $v_3 = (-1,1,2)$

Then $\langle v_1, v_2 \rangle = 1(1) + 1(-1) + 0(1) = 0$

$\langle v_1, v_3 \rangle = 1(-1) + 1(1) + 0(2) = 0$

$\langle v_2, v_3 \rangle = 1(-1) + (-1)(1) + 1(2) = 0$

Now, $\|v_1\| = \sqrt{(1)^2 + (1)^2 + (0)^2} = \sqrt{2} \neq 1$

$\|v_2\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3} \neq 1$

$\|v_3\| = \sqrt{(-1)^2 + 1^2 + (2)^2} = \sqrt{6}$

If we normalize each of the vectors v_1, v_2, v_3 the set an orthonormal set

$\{ \frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{3}}(1,-1,1), \frac{1}{\sqrt{6}}(-1,1,2) \}$

Example-3

For any integer n let $f_n(t) = e^{int}$ where $0 \le t \le 2\pi$

Let $S = \{ f_n : n \text{ is any integer} \}$

$$\langle f_m, f_n \rangle = \frac{1}{2\pi}\int_0^{2\pi} f_m(t)\,\overline{f_n(t)}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{imt}\, e^{-int}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)t}\,dt$$

$$= \frac{1}{2\pi}\left.\frac{e^{i(m-n)t}}{i(m-n)}\right|_0^{2\pi} \quad \text{if } m \ne n$$

$$= \frac{1}{2\pi\, i(m-n)}\Big(\cos(m-n)t + i\sin(m-n)t\Big)\Big|_0^{2\pi}$$

if $m \ne n$

$$= 0 \quad \text{if } m \ne n$$

and $\langle f_n, f_n \rangle = \frac{1}{2\pi}\int_0^{2\pi} f_n(t)\,\overline{f_n(t)}\,dt$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{int}\, e^{-int}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 1 \, dt$$

$$= \frac{1}{2\pi}\, 2\pi = 1 \quad \text{for each } n$$

So S is an orthonormal set.

Ex - 6.1

(9) (a) (c) $\beta = \{v_1, v_2, \ldots v_n\}$

Since $\langle x, z \rangle = 0 \;\forall\; z \in \beta$

$\Rightarrow \langle x, v_i \rangle = 0$ for each $i = 1, 2, \ldots n$

let $v \in V$

Then $v = \sum_{i=1}^{n} a_i v_i$, $a_1, a_2, \ldots a_n$ are scalars

Now, $\langle x, v \rangle = \langle x, \sum_{i=1}^{n} a_i v_i \rangle$

$= \bar{a}_1 \langle x, v_1 \rangle + \bar{a}_2 \langle x, v_2 \rangle + \ldots + \bar{a}_n \langle x, v_n \rangle$

$= \bar{a}_1 (0) + \bar{a}_2 (0) + \ldots + \bar{a}_n (0)$

$= 0$ for all $v \in V$

$\Rightarrow \langle x, x \rangle = 0$

$\Rightarrow x = 0$

(b) let $\langle x, z \rangle = \langle y, z \rangle \;\forall\; z \in \beta$

$\Rightarrow \langle x, z \rangle - \langle y, z \rangle = 0 \;\forall\; z \in \beta$

$\Rightarrow \langle x - y, z \rangle = 0 \;\forall\; z \in \beta$

$\Rightarrow \langle x - y, v_i \rangle = 0$ for each $i = 1, 2, \ldots n$

let $v \in V$

Then $v = \sum_{i=1}^{n} a_i v_i$, $a_1, a_2, \ldots a_n$ are scalars

Now,

$\langle x-y, u\rangle = \langle x-y, \sum_{i=1}^{n} \alpha_i u_i\rangle$

$= \bar{\alpha}_1 \langle x-y, u_1\rangle + \bar{\alpha}_2 \langle x-y, u_2\rangle + \dots$

$+ \bar{\alpha}_n \langle x-y, u_n\rangle$

$= \bar{\alpha}_1 (0) + \bar{\alpha}_2 (0) + \dots + \bar{\alpha}_n (0)$

$= 0 \ \forall\, u \in V$

$\Rightarrow \langle x-y, u\rangle = 0$

$\Rightarrow x-y = 0$

$\Rightarrow x = y$

(2) We are given that

In $C[0,1]$

$f(t) = t, \quad g(t) = e^t$

defined as

$\langle f,g\rangle = \int_0^1 f(t)\, g(t)\, dt$

So, $\langle f,g\rangle = \int_0^1 t e^t \, dt$

$= \left[t e^t - \int e^t \, dt \right]_0^1$

$= \left[t e^t - e^t \right]_0^1$

$= [\cancel{e} - \cancel{e} - 0 + e^0]$

$\boxed{\langle f,g\rangle = 1} = \langle g,f\rangle$ —— (1)

Now, we know that

$\|f\| = \sqrt{\langle f,f\rangle}$

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So, $\langle f, f\rangle = \int_0^1 t^2\, dt$

$= \left[\frac{t^3}{3}\right]_0^1$

$= 1/3$

So $\|f\| = \sqrt{1/3}$

$\boxed{\|f\| = \frac{1}{\sqrt{3}}}$

Also, $\|g\| = \sqrt{\langle g, g\rangle}$

So, $\langle g, g\rangle = \int_0^1 e^{2t}\, dt$

$= \left[\frac{e^{2t}}{2}\right]_0^1$

$= \frac{e^2 - 1}{2}$

So, $\boxed{\|g\| = \frac{\sqrt{e^2 - 1}}{\sqrt{2}}}$

Also, $\|f+g\| = \sqrt{\langle f+g, f+g\rangle}$

So, $\langle f+g, f+g\rangle$

$= \langle f+g, f\rangle + \langle f+g, g\rangle$

$= \langle f, f\rangle + \langle g, f\rangle + \langle f, g\rangle + \langle g, g\rangle$

$= \langle f, f\rangle + \langle f, g\rangle + \langle f, g\rangle + \langle g, g\rangle$ (from - (i))

$= \langle f, f\rangle + 2\langle f, g\rangle + \langle g, g\rangle$

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$= \frac{1}{3} + 2 + \frac{e^2-1}{2}$

$= \frac{2+12+3e^2-3}{6}$

$= \frac{3e^2+11}{6}$

$\therefore \boxed{\|f+g\| = \sqrt{\frac{3e^2+11}{6}}}$

(2) We are given that

$x = (2, 1+i, i)$ and $y = (2-i, 2, 1+2i)$

$\forall x, y \in C^3$

We know that

$\langle x, y\rangle = \sum a_i \bar{b_i}$

$= 2(2+i) + (1+i)2 + i(1-2i)$

$= 4 + 2i + 2 + 2i + i + 2$

$\boxed{\langle x, y\rangle = 8 + 5i}$

Now,

$\|x\| = \sqrt{\langle x, x\rangle}$

~~First~~

$\langle x, x\rangle = \sum a_i \bar{a_i}$

$= 2\times2 + (1+i)(1-i) + i \times -i$

$= 4 + 1 + 1 + 1$

$= 7$

$\therefore \boxed{\|x\| = \sqrt{7}}$

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Also,

$\|y\| = \sqrt{\langle y, y\rangle}$

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firstly, $\langle y, y\rangle = \sum b_i \bar{b}_i$

$= (2-i)(2+i) + 2\times 2 + (1+2i)(1-2i)$

~~$= 4+1+4+1+2$~~

$= 14$

$$\boxed{\therefore \|y\| = \sqrt{14}}$$

Also, $\|x+y\| = \sqrt{\langle x+y, x+y\rangle}$

firstly, $\langle x+y, x+y\rangle$

$= \langle x+y, x\rangle + \langle x+y, y\rangle$

$= \langle x, x\rangle + \langle y, x\rangle + \langle x, y\rangle + \langle y, y\rangle$

~~$= \langle x, x\rangle + 2\langle x, y\rangle + \langle y, y\rangle$~~

~~$= 7 + 2(8+5i) + 14$~~

~~$= 37 + 10i$~~

$= \langle x, x\rangle + \overline{\langle x, y\rangle} + \langle x, y\rangle + \langle y, y\rangle$

$= 7 + 8 - \cancel{8i} + 8 + \cancel{8i} + 14$

$= 37$

$$\therefore \boxed{\|x+y\| = \sqrt{37}}$$

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(15) $\|x+y\|^2 = \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2$

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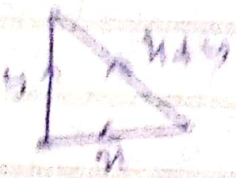
Since x and y are orthogonal

$\therefore \langle x,y\rangle = 0$

So $\overline{\langle x,y\rangle} = 0$

Then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$



(16) $\left\| \sum_{j=1}^{k} a_j v_j \right\|^2$

$= \left\langle \sum_{i=1}^{k} a_i v_i , \sum_{j=1}^{k} a_j v_j \right\rangle$

$= \sum_{i=1}^{k} \sum_{j=1}^{k} a_i \overline{a_j} \langle v_i, v_j\rangle$

$= \sum_{j=1}^{k} a_j \overline{a_j} \langle v_j, v_j\rangle \quad (\because \langle v_i, v_j\rangle = 0$ for $i \neq j)$

$= \sum_{j=1}^{k} |a_j|^2 \|v_j\|^2$

(17) let $x \in \ker T$

$\Rightarrow T(x) = 0$

$\Rightarrow \|T(x)\| = 0$

$\Rightarrow \|x\| = 0$

$\Rightarrow x = 0$

$\therefore \ker T = \{0\}$

So, T is one-to-one

(G.P)

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Definition

Let V be an inner product space. A subset of V is an orthonormal basis for V if it is an ordered basis that is orthonormal.

Example (1)

In F^n the standard ordered basis

$\beta = \{e_1=(1,0,\dots,0), e_2=(0,1,0,\dots,0),\dots, e_n=(0,0,\dots,1)\}$

is an orthonormal basis for F^n.

Solution

Now $\langle e_i, e_j\rangle = 0(0)+0(0)+\dots+1(0)+\dots+0(1)+\dots$
$+0(0)$
$= 0$ for $i\neq j$

$\|e_i\|^2 = \langle e_i, e_i\rangle = 0^2+0^2+\dots+1^2+\dots+0^2$
$= 1$ for each $i=1,2,\dots,n$

So β is an orthonormal set.

Thus β is an orthonormal basis for F^n.

Example 2

The set $\beta = \left\{\left(\frac{1}{\sqrt5},\frac{2}{\sqrt5}\right),\left(\frac{2}{\sqrt5},\frac{-1}{\sqrt5}\right)\right\}$

is an orthonormal basis for R^2.

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Solution

Let $v_1 = \left(\frac{1}{\sqrt5}, \frac{2}{\sqrt5}\right)$, $v_2 = \left(\frac{2}{\sqrt5}, -\frac{1}{\sqrt5}\right)$

Let $a, b \in R$ s.t.

$av_1 + bv_2 = 0$

$\Rightarrow a\left(\frac{1}{\sqrt5}, \frac{2}{\sqrt5}\right) + b\left(\frac{2}{\sqrt5}, \frac{-1}{\sqrt5}\right)$

$\Rightarrow \left(\frac{a+2b}{\sqrt5}, \frac{2a-b}{\sqrt5}\right) = (0,0)$

$\Rightarrow a+2b=0$

$2a-b=0$

$\Rightarrow b=2a$

So, $a+4a=0$

$\Rightarrow 5a=0$

$\Rightarrow a=0$

$\Rightarrow b=2a=0$

So, $\beta = \{v_1, v_2\}$ is a linearly independent set in R^2 and therefore it is an ordered basis for R^2 ($\because \dim(R^2)=2$)

Now, $\langle v_1, v_2 \rangle = \frac{1}{\sqrt5}\left(\frac{2}{\sqrt5}\right) + \frac{2}{\sqrt5}\left(-\frac{1}{\sqrt5}\right) = 0$

$\|v_1\| = \sqrt{\left(\frac{1}{\sqrt5}\right)^2 + \left(\frac{2}{\sqrt5}\right)^2} = \sqrt{\frac{1+4}{5}} = 1$

Theorem 6.3

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$$\|u_1\| = \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{-1}{\sqrt{5}}\right)^2} = \sqrt{\frac{1+4}{5}} = 1$$

$\therefore$ β is an orthonormal basis for R^2.

Theorem 6.3

Let V be an inner product space and $S = \{u_1, u_2, \ldots, u_k\}$ be an orthogonal subset of V consisting of nonzero vectors. If $y \in \text{span}(S)$, then

$$y = \sum_{i=1}^{k} \frac{\langle y, u_i\rangle}{\|u_i\|^2} u_i$$

Proof. Since $y \in \text{Span}(S)$

$\therefore y = \sum_{i=1}^{k} a_i u_i$, where $a_1, a_2, \ldots, a_k$ are scalars

Now, for each $j = 1, \ldots, k$

$$\langle y, u_j\rangle = \left\langle \sum_{i=1}^{k} a_i u_i, u_j \right\rangle$$

$$= \sum_{i=1}^{k} a_i \langle u_i, u_j\rangle$$

$$= a_j \langle u_j, u_j\rangle \quad \left(\because \langle u_i, u_j\rangle = 0 \text{ for } i \neq j\right)$$

$$= a_j \|u_j\|^2$$

then $v_j \neq 0$ for each $j = 1, \ldots, k$

$\therefore \|v_j\|^2 \neq 0$ for each $j = 1, \ldots, k$

So $a_j = \dfrac{\langle y, v_j \rangle}{\|v_j\|^2}$ for $j = 1, \ldots, k$

Thus $y = \sum_{j=1}^{k} a_j v_j$

$$= \sum_{j=1}^{k} \frac{\langle y, v_j \rangle}{\|v_j\|^2} v_j$$

Corollary 1

Let V be an inner product space and $S = \{v_1, v_2, \ldots, v_k\}$ be an orthonormal subset of V. If $y \in \operatorname{span}(S)$, then

$$y = \sum_{j=1}^{k} \langle y, v_j \rangle v_j$$

Proof Since $y \in \operatorname{span}(S)$

$\therefore y = \sum_{i=1}^{k} a_i v_i$, where $a_1, a_2, \ldots, a_k$ are scalars

Now for each $j = 1, \ldots, k$

$$\langle y, v_j \rangle = \left\langle \sum_{i=1}^{k} a_i v_i, v_j \right\rangle$$

$$= \sum_{i=1}^{k} a_i \langle v_i, v_j \rangle$$

$$= a_j \langle v_j, v_j \rangle \quad (\because \ldots)$$

$$= a_j \|v_j\|^2$$

$= a_j \quad [\because \|u_j\|^2 = 1]$

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$\Rightarrow y = \sum_{i=1}^{k} \langle y, u_i \rangle u_i$

Corollary 2

Let V be an inner product space and S be an orthogonal subset of V consisting of non-zero vectors. Then S is linearly independent.

Proof

Let $v_1, v_2 \dots v_k \in S$

and $\sum_{i=1}^{k} a_i v_i = 0$

where $a_1, a_2 \dots, a_k$ are scalars

Now, for each $j = 1, 2 \dots k$

$$\left\langle \sum_{i=1}^{k} a_i v_i, v_j \right\rangle = 0$$

$$\Rightarrow \sum_{i=1}^{k} a_i \langle v_i, v_j \rangle = 0$$

$$\Rightarrow a_j \langle v_j, v_j \rangle = 0 \quad \left(\because \langle v_i, v_j \rangle = 0 \text{ for } i \neq j\right)$$

Since $v_j \neq 0$ for each $j = 1, 2 \dots k$

$\langle v_j, v_j \rangle \neq 0$ for each $j = 1, 2 \dots k$

So, $a_j = 0$ for each $j = 1, 2 \dots k$

~~$= a_j$~~ ... linearly independent.

Corollary 1 (Not in book)

Let V be an inner product space & S be an orthonormal subset of V. Then S is linearly independent.

Proof

Let $v_1, v_2, \ldots, v_k \in S$

& $\sum_{i=1}^{k} a_i v_i = 0$

where $a_1, a_2, \ldots, a_k$ are scalars.

Now for each $j = 1, 2, \ldots, k$

$$\left\langle \sum_{i=1}^{k} a_i v_i, v_j \right\rangle = 0$$

$$\Rightarrow \sum_{i=1}^{k} a_i \langle v_i, v_j \rangle = 0$$

$$\Rightarrow a_j \langle v_j, v_j \rangle = 0 \quad (\because \langle v_i, v_j \rangle = 0 \text{ for } i \neq j)$$

for each $j = 1, 2, \ldots, k$

But $\langle v_j, v_j \rangle = 1$ for each $j = 1, 2, \ldots, k$

So, $a_j = 0$ for each $j = 1, 2, \ldots, k$

So, S is linearly independent.

Example :

The orthonormal set

$$\left\{ \frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{3}}(1,-1,1), \frac{1}{\sqrt{6}}(-1,1,2) \right\}$$

in R^3. So it is a linearly independent set in R^3.

Solution

Since $\dim(R^3) = 3$,

$$\therefore \left\{ \overset{v_1}{\frac{1}{\sqrt{2}}(1,1,0)}, \overset{v_2}{\frac{1}{\sqrt{3}}(1,-1,1)}, \overset{v_3}{\frac{1}{\sqrt{6}}(-1,1,2)} \right\}$$

is an orthonormal basis for R^3.

Let $x = (2,1,3) \in R^3$.

then $x = a_1v_1 + a_2v_2 + a_3v_3$

$\therefore a_1 = \langle x, v_1 \rangle$

$= \frac{1}{\sqrt{2}}(2(1) + 1(1) + 3(0))$

$= 3/\sqrt{2}$

$a_2 = \langle x, v_2 \rangle$

$= \frac{1}{\sqrt{3}}(2(1) + 1(-1) + 3(1))$

$= \frac{4}{\sqrt{3}}$

$a_3 = \langle x, v_3 \rangle$

$= \frac{1}{\sqrt{6}}(2(-1) + 1(1) + 3(2)) = \frac{5}{\sqrt{6}}$

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$\therefore (2,1,3) = \frac{3}{\sqrt{2}}\frac{1}{\sqrt{2}}(1,1,0) + \frac{4}{\sqrt{3}}\frac{1}{\sqrt{3}}(1,-1,1) + \frac{5}{\sqrt{6}}\frac{1}{\sqrt{6}}(-1,1,2)$

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$$= \frac{3}{2}(1,1,0) + \frac{4}{3}(1,-1,1) + \frac{5}{6}(-1,1,2).$$

— x —

Theorem 6.4

Let V be an inner product space and $S = \{w_1, w_2 \ldots w_n\}$ be a linearly independent subset of V. Define $S' = \{v_1, v_2 \ldots v_n\}$, where $v_1 = w_1$ and

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \quad \text{for } 2 \le k \le n$$

Then S' is an orthogonal set of nonzero vectors such that $\text{Span}(S') = \text{span}(S)$.

Proof.

We shall prove this result by induction on n, the number of vectors in the set S.

for $k = 1, 2 \ldots n$ let

$S_k = \{w_1, w_2 \ldots w_k\}$

If $n = 1$, then result holds by taking $S_1' = S_1$ i.e. $v_1 = w_1$

If $n = 2$,

let $v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{||v_1||^2} v_1$

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~~$= w_2 - \frac{\langle w_2, w_1 \rangle}{||w_1||^2} w_1$~~

Now, $\langle v_2, v_1 \rangle = \langle w_2 - \frac{\langle w_2, w_1 \rangle}{||w_1||^2} w_1, w_1 \rangle$

$= \langle w_2, w_1 \rangle - \frac{\langle w_2, w_1 \rangle}{\cancel{||w_1||^2}} \cancel{\langle w_1, w_1 \rangle}$

$= \langle w_2, w_1 \rangle - \langle w_2, w_1 \rangle$

$= 0$

$\therefore S_2' = \{v_1, v_2\}$ is an orthogonal set of nonzero vectors

Now $\text{span}(S_2') = \{a v_1 + b v_2 : a, b \text{ are scalar}\}$

$= \{a w_1 + b(w_2 - \frac{\langle w_2, w_1 \rangle}{||w_1||^2} w_1) : a, b \text{ are scalar}\}$

$= \{(a - \frac{b \langle w_2, w_1 \rangle}{||w_1||^2}) w_1 + b w_2 : a, b \text{ are scalar}\}$

$= \text{span}(S_2)$

So, the result holds for $n=2$

Suppose that the result holds for $n = k-1$

i.e. the set $S_{k-1}' = \{v_1, v_2 \text{ ——— } v_{k-1}\}$ is an orthogonal set of nonzero vectors are

$Span(S'_{k-1}) = Span(S_{k-1})$

We shall prove that the set $S_k' = \{u_1, u_2, \ldots, u_{k-1}, u_k\}$ is an orthogonal set of non-zero vectors and $Span(S_k') = Span(S_k)$, where u_k is obtained from S'_{k-1} by

$$u_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, u_j \rangle}{\|u_j\|^2} u_j$$

Now, $u_k \neq 0$

$\because$ If $u_k = 0$, then

$$w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, u_j \rangle}{\|u_j\|^2} u_j = 0$$

$$\Rightarrow w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, u_j \rangle}{\|u_j\|^2} \in Span(S'_{k-1})$$

$$= Span(S_{k-1})$$

$\Rightarrow \{w_1, w_2, \ldots w_{k-1}, w_k\}$ is a linearly dependent set

$\Rightarrow S_k$ is linearly dependent

which is a contradiction

($\because S_k$ is a linearly independent set).

Now,

for $j = 1, 2, \ldots, k-1$

$\langle u_k, v_j \rangle$

$$= \left\langle w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, u_j \rangle}{\|u_j\|^2} u_j, v_j \right\rangle$$

$$= \langle w_k, u_j \rangle - \sum_{j=1}^{k-1} \frac{\langle w_k, u_j \rangle}{\|u_j\|^2} \langle u_j, u_j \rangle$$

$$= \langle w_k, u_j \rangle - \frac{\langle w_k, u_j \rangle}{\cancel{\|u_j\|^2}} \cancel{\langle u_j, u_j \rangle} \quad (\because \langle u_i, u_j \rangle = 0 \text{ for } i \neq j)$$

$$= 0$$

Thus S_k' is an orthogonal set of non-zero vectors.

Since, $u_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, u_j \rangle}{\|u_j\|^2} u_j$

$\therefore \text{Span}(S_k') \subseteq \text{Span}(S_k)$ $\left(\because \text{ if } x \in \text{Span}(S_k')\right)$

$$\Rightarrow x = \sum_{j=1}^{k} a_j u_j$$

$$= \sum_{j=1}^{k-1} a_j u_j + a_k u_k$$

$$= \sum_{j=1}^{k-1} a_j u_j + a_k \left(w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, u_j \rangle}{\|u_j\|^2} u_j \right.$$

$$= \sum_{j=1}^{k-1} \left(a_j - a_k \frac{\langle w_k, u_j \rangle}{\|u_j\|^2} \right) u_j + a_k w_k$$

Now $\sum_{j=1}^{k-1} \left(a_j - a_k \frac{\langle w_k, u_j \rangle}{\|u_j\|^2} \right) u_j \in \mathrm{Span}(S'_{k-1}) = \mathrm{Span}(S_{k-1})$

So, $u \in \mathrm{Span}(S_k)$

Since, S_k is linearly independent

$\therefore \dim(\mathrm{Span}(S'_k)) = \dim(\mathrm{Span}(S_k)) = k$

So, $\mathrm{Span}(S_k') = \mathrm{Span}(S_k)$.

—x—

Remark

The construction of $\{v_1, v_2, \ldots, v_k\}$ in the above theorem is called the ~~Gram~~ Gram-Schmidt process.