# Linear Algebra (GE-2)

Vikendra Singh

# Lecture 1

**Vector Space:** Let $V$ be an arbitrary nonempty set of objects, together with two operations namely **addition** (denoted as $\oplus$) and **scalar multiplication**(denoted as $\odot$), is said to be a (real) vector space if for every $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $V$ and for every $a, b \in \mathbb{R}$ the following properties hold:

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3. $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w})$ (Associativity)
4. There exists an element $0 \in V$, called a **zero vector**, such that $\mathbf{u} \oplus 0 = \mathbf{u}$ (Existence of additive identity)

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The objects of a vector space $V$ are called **vectors**.

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The vector space $V = \{0\}$ is called the zero (trivial) vector space.

**Example 1:** The set $\mathbb{R}$ of real numbers is a vector space with respect to the following operations:

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for all $a, \mathbf{u}, \mathbf{v} \in \mathbb{R}$.

**Question:** Does the set $\mathbb{R}^+$ of positive real numbers form a vector space under the above defined vector addition $\oplus$ and scalar multiplication $\odot$?

**Example 2:** The set $\mathbb{R}^+$ of a positive real numbers is a vector space with respect to the following operations:

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- $a \odot \mathbf{u} = \mathbf{u}^a$ (scalar multiplication)

for all $a \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^+$.

**Example 3:** The set $\mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$ is a vector space with respect to the following vector addition $\oplus$ and scalar multiplication $\odot$:

- $(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$

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**Question:** Does $\mathbb{R}^2$ form a vector space under the above defined vector addition and the following scalar multiplication

$$a \odot (x_1, x_2) = (0, ax_2)$$

for all $a \in \mathbb{R}$ and $(x_1, x_2) \in \mathbb{R}^2$.

**Soln. of Example 3:** Let $\mathbf{u} = (x_1, x_2)$, $\mathbf{v} = (y_1, y_2)$ and $\mathbf{w} = (z_1, z_2) \in \mathbb{R}^2$ and $a, b \in \mathbb{R}$.

1. Closure Property:
   $\mathbf{u} \bigoplus \mathbf{v}$

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4. Existence of additive identity (zero vector):

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5. Existence of additive inverse:

$$= (x_1, x_2) \oplus (y_1 + z_1, y_2 + z_2)$$
$$= (x_1, x_2) \oplus ((y_1, y_2) \oplus (z_1, z_2))$$
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4. Existence of additive identity (zero vector): For any $\mathbf{u} = (x_1, x_2) \in \mathbb{R}^2$ there exists $\mathbf{0} = (0, 0) \in \mathbb{R}^2$ such that

$$\mathbf{u} \oplus \mathbf{0} = (x_1, x_2) \oplus (0, 0) = (x_1 + 0, x_2 + 0)$$
$$= (x_1, x_2)$$
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5. Existence of additive inverse: For each $\mathbf{u} = (x_1, x_2) \in \mathbb{R}^2$ there exists $-\mathbf{u} = (-x_1, -x_2)$ in $\mathbb{R}^2$ such that

$$= (x_1, x_2) \oplus (y_1 + z_1, y_2 + z_2)$$
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$$
\begin{aligned}
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(associativity of $\mathbb{R}$ under multiplication)

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10. $1 \odot \mathbf{u} =$

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Thus $\mathbb{R}^2$ is vector space under usual vector addition and scalar multiplication.

**Exercise:** Show that the set

$$\mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$$

is a vector space with respect to the following vector addition $\oplus$ and scalar multiplication $\odot$:

- $(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 - 2)$

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- $(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 - 2)$
- $a \odot (x_1, x_2) = (ax_1 + a - 1, ax_2 - 2a + 2)$

**Example 4:** Consider the set

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \ : \ x_i \in \mathbb{R}\}.$$

For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $a \in \mathbb{R}$, define

**Example 4:** Consider the set

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Then $\mathbb{R}^n$ is a vector space with respect to $\oplus$ and $\odot$.

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$$\mathbf{u} \oplus \mathbf{v} = (x_1, x_2, \ldots, x_n) \oplus (y_1, y_2, \ldots, y_n)$$

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Then $\mathbb{R}^n$ is a vector space with respect to $\oplus$ and $\odot$.

**Example 5:** The set

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of all $m \times n$ matrices with real entries is a vector space with respect to the following operations:

- $[a_{ij}]_{m\times n} \oplus [b_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n}$ (vector addition)
- $a \odot [a_{ij}]_{m\times n} = [aa_{ij}]_{m\times n}$ (scalar multiplication)

for all $a \in \mathbb{R}$ and $[a_{ij}]_{m\times n}, [b_{ij}]_{m\times n} \in M_{mn}$.

**Theorem 4.1.1:** Let $V$ be a vector space. Then for every $\mathbf{u} \in V$ and $k \in \mathbb{R}$, we have

- $k0_V = 0_V$
- $0\mathbf{u} = 0_V$
- $(-1)\mathbf{u} = -\mathbf{u}$
- If $k\mathbf{u} = 0_V$, then $k = 0$ or $\mathbf{u} = 0_V$.

# Lecture 2

# Subspaces

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**Definition:** A nonempty subset $W$ of a vector space $V$ is said to be a subspace of $V$ if $W$ is itself a vector space with respect to the same operations (vector addition and scalar multiplication) of $V$.

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**Note that** every vector space $V$ has at least two subspaces: $\{0\}$ and $V$ itself.

# Subspaces

**Definition:** A nonempty subset $W$ of a vector space $V$ is said to be a subspace of $V$ if $W$ is itself a vector space with respect to the same operations (vector addition and scalar multiplication) of $V$.

**Note that** every vector space $V$ has at least two subspaces: $\{0\}$ and $V$ itself. The subspace $\{0\}$ is known as zero (trivial) subspace.

**Example:** The set

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**Question:** Does the set

$$W = \{(x, y) \in \mathbb{R}^2 \mid x \neq y\}$$

form a subspace of $\mathbb{R}^2$?

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**Remark:** If $W$ is a subspace of a vector space $V$, then $0 \in W$.

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- $W_5 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.

**Exercise:** Examine whether the following sets are subspaces of the vector space $M_{22}$

- $W_1 = \{A \in M_{22} \mid A \text{ is singular}\}$.
- $W_2 = \{A \in M_{22} \mid A \text{ is nonsingular}\}$.
- $W_4 = \{A \in M_{22} \mid A \text{ is symmetric}\}$.
- $W_5 = \{A \in M_{22} \mid A^2 = A\}$.

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- their union $W_1 \cup W_2$ **need not** be a subspace of $V$.
- $W_1 \cup W_2$ is subspace of $V$ if and only if either $W_1 \subset W_2$ or $W_2 \subset W_1$.
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# Lecture 3

**Linear combination:** Let $V$ be a vector space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r \in V$. Then a vector $\mathbf{w} \in V$ is said to be a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ if

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Thus, $(3,4)$ is a linear combination of $(1,1)$ and $(1,2)$ also.

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**Exercise:** Let $V = \mathbb{R}^3$ and $S = \{(1,0,0),(0,1,0)\}$.

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**Solution:**

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Clearly, $(3,2,0) \in$ span$(S)$ but $(2,5,1) \notin$ span$(S)$.

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**Theorem** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a nonempty subset of a vector space $V$. Then

- span($S$) is a subspace of $V$.
- span($S$) is the smallest subspace of $V$ containing $S$.

Convention: span($\emptyset$) $= \{0\}$.

**Exercise:** Determine whether the vectors $\mathbf{v}_1 = (1, 2, 3), \mathbf{v}_2 = (2, 0, 0)$ and $\mathbf{v}_3 = (-2, 1, 0)$ span the vector space $\mathbb{R}^3$.

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Let $(a, b, c)$ be an arbitrary element of $\mathbb{R}^3$. We must check whether $(a, b, c)$ belongs to span$(S)$ or not i.e. whether there exists $k_1, k_2, k_3 \in \mathbb{R}$ such that

$$(a, b, c) = k_1(1, 2, 3) + k_2(2, 0, 0) + k_3(-2, -1, 0)$$

This is equivalent to check whether the system of equations

$$\begin{aligned} k_1 + 2k_2 - 2k_3 &= a \\ 2k_1 - k_3 &= b \\ 3k_1 &= c \end{aligned}$$

is consistent for any $a, b, c \in \mathbb{R}$.

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**Exercise** Determine whether the vectors $\mathbf{v}_1 = (3, 2, 4), \mathbf{v}_2 = (-3, -1, 0), \mathbf{v}_3 = (0, 1, 4)$ and $\mathbf{v}_4 = (0, 2, 8)$ span the vector space $\mathbb{R}^3$.

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**Hint:** By the similar argument, used in previous exercise, one should check whether the system of equations

$$\begin{aligned} 3k_1 - 3k_2 &= a \\ 2k_1 - k_2 + k_3 + 2k_4 &= b \\ 4k_1 + 4k_3 + 8k_4 &= c \end{aligned}$$

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Now show that the reduced row echelon form of the augmented matrix

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$$\begin{bmatrix} 3 & -3 & 0 & 0 & a \\ 2 & -1 & 1 & 2 & b \\ 4 & 0 & 4 & 8 & c \end{bmatrix} \text{ is } \begin{bmatrix} 1 & 0 & 1 & 2 & b - \frac{a}{3} \\ 0 & 1 & 1 & 2 & b - \frac{2a}{3} \\ 0 & 0 & 0 & 0 & 4a - 12b + 3c \end{bmatrix}$$

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**Note that** the vector $(0, 0, 1) \in \mathbb{R}^3$ but it is **not in** span$(S)$.

# Lecture 4

# Linear Independence

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**Definition:** A subset $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of a vector space $V$ is said to be linearly dependent (LD) if there exist real numbers $a_1, a_2, \ldots, a_n$ not all zero such that

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Then

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**Example:** Show that

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**Solution:** Let $a, b, c \in \mathbb{R}$ such that

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To solve above homogenous system, write augmented matrix

$$[A \ \ \mathbf{0}] = \begin{bmatrix} 3 & -5 & 2 & 0 \\ 1 & -2 & 2 & 0 \\ -1 & 2 & -1 & 0 \end{bmatrix}$$

reduced row echelon form of $[A \;\; \mathbf{0}]$ is

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**Exercise:** For a given vector space $V$ and a given subset $S$ of $V$, check the linear independence of $S$ in the following:

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**Exercise:** Examine the linear independence of a subset $S = \{(2, -5, 1), (1, 1, -1), (0, 2, -3), (2, 2, 6)\}$ of $\mathbb{R}^3$.

# Lecture 5

# Coordinates and Basis

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**Definition:** A finite subset $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ of a vector space $V$ is said to be a basis of $V$ if

1. $S$ is LI, and
2. span$(S) = V$.

## Examples

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- The subset $S = \{(1,0,0),(0,1,0),(0,0,1)\}$, also denoted by $\{e_1, e_2, e_3\}$, is a basis of $\mathbb{R}^3$ as it is LI and span$(S) = \mathbb{R}^3$.

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- The subset $S = \{(1,0,0),(0,1,0),(0,0,1)\}$, also denoted by $\{e_1, e_2, e_3\}$, is a basis of $\mathbb{R}^3$ as it is LI and span$(S) = \mathbb{R}^3$. The subset $S$ is called the **standard basis** of $\mathbb{R}^3$.

Analogously, $S = \{e_1, e_2, \ldots, e_n\}$ be a standard basis of $\mathbb{R}^n$, where $e_i$ is a vector of $\mathbb{R}^n$ such that its $i^{\text{th}}$ component is $1$ and remaining components are $0$.

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**Example:** Show that the vectors $\mathbf{v}_1 = (1,2,1)$, $\mathbf{v}_2 = (2,9,0)$ and $\mathbf{v}_3 = (3,3,4)$ form a basis of $\mathbb{R}^3$.

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Verify that $S$ is LI and span$(S) = M_{22}$.

**Theorem:** If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space $V$, then every vector $\mathbf{v}$ in $V$ can be expressed in the form $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ in exactly one way.

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then the scalars $c_1, c_2, \ldots, c_n$ are called **coordinates** of $\mathbf{v}$ relative to the basis $S$.

The vector $(c_1, c_2, \ldots, c_n) \in \mathbb{R}^n$ constructed from these coordinates is called the **coordinate vector of v relative** to $S$; it is denoted by

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This leads to solve the system of equations

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# Lecture 6

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**Example:** The vector spaces $\mathbb{R}^n, P_n$ and $M_{mn}$ are finite dimensional, whereas the vector space $P_\infty$ is infinite dimensional.

**Theorem:** Let $V$ be a finite dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis

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**Theorem:** All bases for a finite dimensional vector space have the same number of elements.

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The dimension of the zero vector space $\{0\}$ is defined to be zero.

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**Theorem:** Let $S$ be a nonempty set of vectors in a vector space $V$.

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$$\text{span}(S) = \text{span}(S - \{\mathbf{v}\}).$$

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# Lecture 7

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$$W = \{(-2s, t, s) \mid t, s \in \mathbb{R}\}$$
$$W = \{s(-2, 0, 1) + t(0, 1, 0) \mid t, s \in \mathbb{R}\}$$
$$W = \text{span}\left(\{(-2, 0, 1), (0, 1, 0)\}\right).$$

Note that the set $\{(-2, 0, 1), (0, 1, 0)\}$ is linearly independent (show it).

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$$W = \{(x, y, z) \in \mathbb{R}^3 \mid x + 2z = 0\}.$$

**Solution:** The general solution of the equation $x + 2z = 0$ is given by $\{(-2s, t, s) \mid t, s \in \mathbb{R}\}$. Thus

$$\begin{aligned} W &= \{(-2s, t, s) \mid t, s \in \mathbb{R}\} \\ W &= \{s(-2, 0, 1) + t(0, 1, 0) \mid t, s \in \mathbb{R}\} \\ W &= \text{span}\left(\{(-2, 0, 1), (0, 1, 0)\}\right). \end{aligned}$$

Note that the set $\{(-2, 0, 1), (0, 1, 0)\}$ is linearly independent (show it).

Hence, the subset $\{(-2, 0, 1), (0, 1, 0)\}$ is a basis of $W$ and $\dim(W) = 2$.

Hence, the subset $\{(-2, 0, 1), (0, 1, 0)\}$ is a basis of $W$ and $\dim(W) = 2$.

**Exercise:** Find a basis and the dimension of a subspace $W$ of $P_3$, where

$$W = \{\mathbf{p} \in P_3 \mid \mathbf{p}(2) = 0\}.$$

**Exercise:** Find a basis for the solution space of the following homogenous linear system

$$\begin{aligned} x + 2y - z &= 0 \\ 2x - y + 2z &= 0 \\ 3x + y + z &= 0 \\ 4x + 3y &= 0 \end{aligned}$$

Hence, find the dimension of the solution space.

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Hence, find the dimension of the solution space.

**Hint:** First find the solution set $S$ of given homogenous system of equations and observe that

$$S = \left\{ t \left( \frac{-3}{5}, \frac{4}{5}, 1 \right) \ : \ t \in \mathbb{R} \right\}$$

$$S = \text{span}\left\{\left(\frac{-3}{5}, \frac{4}{5}, 1\right)\right\}$$

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and $\{(\frac{-3}{5}, \frac{4}{5}, 1)\}$ is LI (why?). Thus, $\{(\frac{-3}{5}, \frac{4}{5}, 1)\}$ forms a basis of solution space and $\dim(S) = 1$.

**Exercise:** Let $S = \{(4, 2, 1), (2, 6, -5), (1, -2, 3)\}$ be a subset of vector space $\mathbb{R}^3$.

- Examine the linear independence of $S$.

**Exercise:** Let $S = \{(4,2,1),(2,6,-5),(1,-2,3)\}$ be a subset of vector space $\mathbb{R}^3$.

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- Find dim(span($S$)).

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**Hint:**

- Let

$$a_1(4,2,1)+a_2(2,6,-5)+a_3(1,-2,3) = \mathbf{0} = (0,0,0)$$

**Exercise:** Let $S = \{(4,2,1),(2,6,-5),(1,-2,3)\}$ be a subset of vector space $\mathbb{R}^3$.

- Examine the linear independence of $S$.
- Find dim(span($S$)).

**Hint:**

- Let

$$a_1(4,2,1)+a_2(2,6,-5)+a_3(1,-2,3) = \mathbf{0} = (0,0,0)$$

On solving above system of equations, we get

$$a_1 = -1, a_2 = 1, a_3 = 2$$

implies $S$ is not LI.

- Note that

$$(2, 6, -5) = (4, 2, 1) - 2(1, -2, 3)$$

implies span$(S)$ = span$(S')$, where

$$S' = \{(4, 2, 1), (1, -2, 3)\}.$$

Now, note that $S'$ is LI (Show it).

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Now, note that $S'$ is LI (Show it). Thus $S'$ (a set of two elements) is a basis of span$(S)$ and

$$\text{dim}(\text{span}(S)) = 2.$$

**Theorem:** Let $W$ be a subspace of a finite dimensional vector space $V$. Then

- $W$ is also finite dimensional and dim$W \leq$ dim$V$.
- dim$W =$ dim$V$ if and only if $W = V$.

# Lecture 8

# Subspaces associated with Matrices

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**Definition** Let $A$ be an $m \times n$ matrix.

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- The **column space** of $A$ is the subspace col$(A)$ of $\mathbb{R}^m$ spanned by the column vectors of $A$.
- The **null space** of $A$ is the subspace of $\mathbb{R}^n$ consisting of solutions of the homogenous linear system $A\mathrm{x} = 0$. It is denoted by null$(A)$.

**Exercise:** Find a basis for the null space of

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$$

**Exercise:** Find a basis for the null space of

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**Hint:** Since

$$\text{null}(A) = \{\mathsf{x} : A\mathsf{x} = 0\}$$

On solving right hand side with the above matrix $A$, we get

$$\text{null}(A) = \{(-r-2s-t, -r-s-2t, r, s, t) : r, s, t \in \mathbb{R}\}$$
$$= \text{span}(S), \text{where}$$

$$S = \{(-1,-1,1,0,0), (-2,-1,0,1,0), (-1,-2,0,0,1)\}$$

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Also, show that $S$ is linearly independent.

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Also, show that $S$ is linearly independent. Thus $S$ is a basis for null$(A)$.

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$$= \text{span}(S), \text{ where}$$

$$S = \{(-1, -1, 1, 0, 0), (-2, -1, 0, 1, 0), (-1, -2, 0, 0, 1)\}$$

Also, show that $S$ is linearly independent. Thus $S$ is a basis for null$(A)$. Hence, dim(null$(A)) = 3$.

**Theorem:** If a matrix $R$ is in row echelon form, then the row vector with the leading $1$'s (the nonzero row vectors) form a basis for the row space of $R$,

**Theorem:** If a matrix $R$ is in row echelon form, then the row vector with the leading $1$'s (the nonzero row vectors) form a basis for the row space of $R$, and the column vectors with the leading $1$'s of the row vector form a basis for the column space of $R$.

**Exercise:** Find a basis for the row space and column space of

$$A = \begin{bmatrix} 1 & -3 & 2 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Exercise:** Find a basis for the row space and column space of

$$A = \begin{bmatrix} 1 & -3 & 2 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

**Solution:** Since given matrix is in row echelon form. By Theorem, the set of row vectors

$$\{(1, -3, 2, 4), (0, 1, -1, 0), (0, 0, 1, 3), (0, 0, 0, 1)\}$$

forms a basis of row$(A)$, and the vectors

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{c}_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{c}_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{c}_4 = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

form a basis of col$(A)$.

# Lecture 9

**Exercise:** Find a basis for the row space

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$$

**Exercise:** Find a basis for the row space

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$$

**Solution:** Let $B$ be the RREF of the given matrix. Then find that

$$B = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since $B$ is row equivalent to $A$, we have

$$\text{row}(B) = \text{row}(A).$$

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$$\text{row}(B) = \text{row}(A).$$

Thus, By Theorem, the set of row vectors

$$\{(1,0,1,2,1),(0,1,1,1,2)\}$$

is a basis of row$(A)$.

**Example:** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = (1, 2, 3, -1, 0), \ \mathbf{v}_2 = (3, 6, 8, -2, 0)$$

$$\mathbf{v}_3 = (-1, -1, -3, 1, 1), \ \mathbf{v}_4 = (-2, -3, -5, 1, 1)$$

be a subset of $\mathbb{R}^5$.

**Example:** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = (1, 2, 3, -1, 0), \ \mathbf{v}_2 = (3, 6, 8, -2, 0)$$

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be a subset of $\mathbb{R}^5$. Find a basis for span$(S)$.

**Example:** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = (1, 2, 3, -1, 0),\ \mathbf{v}_2 = (3, 6, 8, -2, 0)$$

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be a subset of $\mathbb{R}^5$. Find a basis for span$(S)$.

**Solution:**

**Example:** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = (1, 2, 3, -1, 0),\ \mathbf{v}_2 = (3, 6, 8, -2, 0)$$

$$\mathbf{v}_3 = (-1, -1, -3, 1, 1),\ \mathbf{v}_4 = (-2, -3, -5, 1, 1)$$

be a subset of $\mathbb{R}^5$. Find a basis for span$(S)$.

**Solution:**

**Step 1:**

$$A = \begin{bmatrix} 1 & 2 & 3 & -1 & 0 \\ 3 & 6 & 8 & -2 & 0 \\ -1 & -1 & -3 & 1 & 1 \\ -2 & -3 & -5 & 1 & 1 \end{bmatrix}$$

## Step 2:

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

## Step 2:

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 & 2 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

## Step 3:

$$B = \{(1, 0, 0, 2, -2), (0, 1, 0, 0, 1), (0, 0, 1, -1, 0)\}$$

is a basis for span$(S)$.

**Theorem :** If $A$ and $B$ are row equivalent matrices, then:

- A given set of column vectors of $A$ forms a basis for the column space of $A$ if and only if the corresponding column vectors of $B$ forms a basis for the column space of $B$.

**Exercise:** Find a basis for the column space

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$$

**Exercise:** Find a basis for the column space

$$A = \begin{bmatrix} 1 & 4 & 5 & 6 & 9 \\ 3 & -2 & 1 & 4 & -1 \\ -1 & 0 & -1 & -2 & -1 \\ 2 & 3 & 5 & 7 & 8 \end{bmatrix}$$

**Solution:** Let $B$ be the RREF of the given matrix. Then find that

$$B = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since First and second column vector of $B$ is a basis for the col$(B)$(Why?).

Since First and second column vector of $B$ is a basis for the col$(B)$(Why?). By Theorem 4.7.6, the set of column vectors

$$\{(1,3,-1,2),(4,-2,0,3)\}$$

is a basis of col$(A)$.

**Example:** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, where

$$\mathbf{v}_1 = (1, 2, -2, 1), \quad \mathbf{v}_2 = (-3, 0, -4, 3)$$

$$\mathbf{v}_3 = (2, 1, 1, -1), \quad \mathbf{v}_4 = (-3, 3, -9, 6)$$

$$\text{and} \quad \mathbf{v}_5 = (9, 3, 7, -6)$$

be a subset of $\mathbb{R}^4$.

**Example:** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, where

$$\mathbf{v}_1 = (1, 2, -2, 1), \quad \mathbf{v}_2 = (-3, 0, -4, 3)$$

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$$\text{and } \mathbf{v}_5 = (9, 3, 7, -6)$$

be a subset of $\mathbb{R}^4$. Find a basis for span$(S)$ consisting all the vectors from $S$.

**Example:** Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$, where

$$\mathbf{v}_1 = (1, 2, -2, 1), \quad \mathbf{v}_2 = (-3, 0, -4, 3)$$

$$\mathbf{v}_3 = (2, 1, 1, -1), \quad \mathbf{v}_4 = (-3, 3, -9, 6)$$

$$\text{and} \quad \mathbf{v}_5 = (9, 3, 7, -6)$$

be a subset of $\mathbb{R}^4$. Find a basis for span$(S)$ consisting all the vectors from $S$.

**Solution:**

$$A = \begin{bmatrix} 1 & -3 & 2 & -3 & 9 \\ 2 & 0 & 1 & 3 & 3 \\ -2 & -4 & 1 & -9 & 7 \\ 1 & 3 & -1 & 6 & -6 \end{bmatrix}$$

$$\text{RREF}(A) = \begin{bmatrix} \color{red}{1} & 0 & 1/2 & 3/2 & 3/2 \\ 0 & \color{red}{1} & -1/2 & 3/2 & -5/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 1/2 & 3/2 & 3/2 \\ 0 & 1 & -1/2 & 3/2 & -5/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The set of vectors corresponding to pivot columns is

$$B = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(1, 2, -2, 1), (-3, 0, -4, 3)\}$$

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 1/2 & 3/2 & 3/2 \\ 0 & 1 & -1/2 & 3/2 & -5/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The set of vectors corresponding to pivot columns is

$$B = \{\mathbf{v}_1, \mathbf{v}_2\} = \{(1, 2, -2, 1), (-3, 0, -4, 3)\}$$

forms a basis for the subspace span$(S)$.

# Lecture 10

# Rank and Nullity of a Matrix

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**Theorem:** The row space and column space of a matrix have the same dimension.

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**Definition:** The common dimension of row$(A)$ and col$(A)$ of a matrix $A$ is called the ***rank*** of $A$ and is denoted by rank$(A)$;

## Rank and Nullity of a Matrix

**Theorem:** The row space and column space of a matrix have the same dimension.

**Definition:** The common dimension of row$(A)$ and col$(A)$ of a matrix $A$ is called the ***rank*** of $A$ and is denoted by rank$(A)$;

- dim(null$(A)$) is called the ***nullity*** of $A$ and it is denoted by nullity$(A)$.

**Result:** For any matrix $A$,

$$\text{rank}(A) = \text{rank}(A^T).$$

**Exercise:** Find the rank and nullity of the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & 4 \\ 2 & 4 & 2 & 0 \\ -1 & -3 & 0 & 5 \end{bmatrix}$$

**Theorem (Dimension Theorem for Matrices):** If $A$ is a matrix with $n$ columns, then

$$\text{rank}(A) + \text{nullity}(A) = n$$

**Theorem:** Let $A$ be an $n \times n$ matrix. The following statements are equivalent:

- $A$ is invertible.
- $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b} \in \mathbb{R}^n$.
- The homogenous system $A\mathbf{x} = 0$ has only the trivial solution.
- The reduced row echelon form of $A$ is $I_n$.
- $A$ is expressible as a product of elementary matrices.
- $\det(A) \neq 0$.
- The column vectors of $A$ are linearly independent.
- The column vectors of $A$ span $\mathbb{R}^n$.

## Theorem: (contd.)

- The column vectors of $A$ form a basis of $\mathbb{R}^n$.
- The row vectors of $A$ are linearly independent.
- The row vectors of $A$ span $\mathbb{R}^n$.
- The row vectors of $A$ form a basis of $\mathbb{R}^n$.
- $A$ has rank $n$.
- $A$ has nullity $0$.

## (Conclusion)

1. Real Vector Spaces
2. Subspaces
3. Span
4. Linear Independence
5. Basis and Dimension
6. Row space, Column Space, and Null Space
7. Rank and Nullity of a Matrix

***Thank You***