Maxwell's Equations and Electromagnetic waves.

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Equation of continuity:

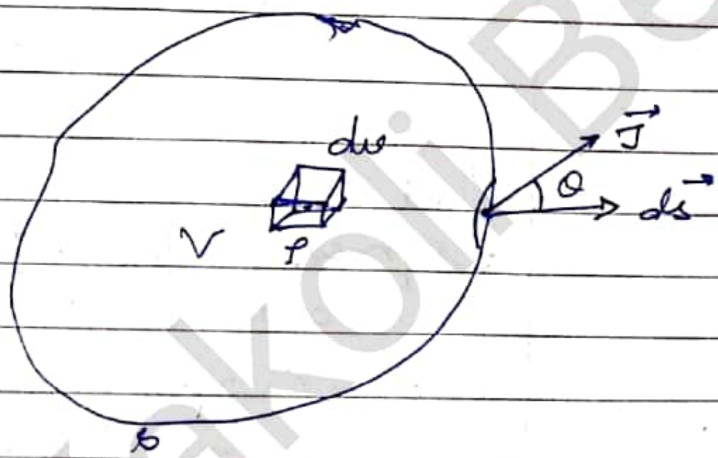
Conservation of charge. (According to principle of conservation of charge the net amount of charge in an isolated system remains constant.) For generality let us assume that the charge density is a function of time. Then the principle of conservation of charge may be stated as follows:

If the net charge crossing a surface bounding a closed volume is not zero, then the charge density within the volume must change with time in such a manner that the time rate of increase of charge within the volume equals the net rate of flow charge into the volume.

This statement of conservation of charge in a medium may be expressed by the equation of continuity which may be derived as follows:

Let S be the surface enclosing a volume V and let $d\vec{s}$ be a small ~~element~~ element of this surface. The direction of $d\vec{s}$ is taken to be that of the outward normal. If $\vec{J}$ is the current density (i.e., current per unit area placed normal of direction of current flow) at a point on surface element $d\vec{s}$, then $\vec{J}\cdot d\vec{s}$ represents the charge per unit time leaving volume V across ds. Therefore the time rate at which charge leaves the volume V ~~across ds. Therefore the~~ bounded by entire surface S is given by

$$\iint_S \vec{J}\cdot d\vec{s}.$$



If q is charge contained in V, then according to charge conservation law the above integral must be equal to $-dq/dt$, where dq/dt represents the time rate of flow of charge into V, thus

$$\iint \vec{J}\cdot d\vec{s} = -\frac{dq}{dt} \quad \text{——} \; (1)$$

But $q = \iiint_V \rho \, dv$

where ρ is the charge density and dv is an element of volume.

Eqn (1) is

$$\iiint_S \vec{J}\cdot d\vec{s} = -\iiint_V \frac{\partial \rho}{\partial t}\, dv \quad \text{——} \; (2).$$

From Gauss divergence

$$\iint_S \vec{J}\cdot d\vec{s} = \iiint_V \operatorname{div} \vec{J}\, dv \quad \text{——} \; (3)$$

Comparing (2) ~~(3)~~ (3).

$$\iiint_V \operatorname{div} \vec{J}\, dv = -\iiint_V \frac{\partial \rho}{\partial t}\, dv$$

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or
$$\iiint_V \left(\text{div}\,\vec{J} + \frac{\partial \rho}{\partial t}\right) dv = 0.$$

Since volume is arbitrary, therefore Integrand must be zero, i.e.,

$$\boxed{\text{div}\,\vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

This is the equation of continuity & expresses the conservation of charge. The current is called stationary if there is no accumulation of charge at any point i.e for stationary current $\frac{\partial \rho}{\partial t} = 0$ at all points.

Therefore the criterion for stationary flow is

$$\boxed{\text{div}\,\vec{J} = \vec{\nabla}\cdot\vec{J} = 0}$$

Displacement current: Maxwell's postulate

From Ampere's circuital law, we have

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 i \quad \text{— (1)}$$

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \int_S \vec{J}\cdot d\vec{s} \quad \text{— (2)}$$

$$\Rightarrow \oint_C \frac{\vec{B}}{\mu_0}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{s}$$

$$\Rightarrow \oint_C \vec{H}\cdot d\vec{l} = \int_S \vec{J}\cdot d\vec{s} \quad \text{— (3)}$$

Using stokes's theorem,

$$\int_S \text{curl}\,\vec{H}\cdot d\vec{s} = \int_S \vec{J}\cdot d\vec{s} \quad \text{— (4)}$$

Since $\vec{B} = \mu_0 \vec{H}$ for free space.

$$\Rightarrow \int_S (\text{curl } \vec{H} - \vec{J}) \cdot d\vec{s} = 0$$

As the surface is arbitrary, therefore integrand must vanish,

$$\text{curl } \vec{H} - \vec{J} = 0$$

$$\Rightarrow \text{curl } \vec{H} = \vec{J} \quad \text{—— (5)}$$

$$\Leftrightarrow \text{Curl } \vec{B} = \mu_0 \vec{J} \quad \text{—— (6)}$$

Let us check for the validity of this equation for time-varying fields.
Since div of curl of any vector is always zero, therefore div curl $\vec{H} = 0$
Then eqn. (5) implies.

$$\text{div } \vec{J} = 0 \quad \text{—— (7)}$$

Now continuity equation is

$$\text{div} \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\Rightarrow \text{div } \vec{J} = -\frac{\partial \rho}{\partial t} \quad \text{—— (8)}$$

According to this equation div $\vec{J} = 0$ only if $\frac{\partial \rho}{\partial t} = 0$ i.e charge density is static.

Thus we conclude that Ampere's eqn (1) is valid only for steady state conditions and is insufficient for the cases of time varying fields. Hence to include time-varying fields Ampere's law must be modified. Maxwell investigated (suggested) that Ampere's equation can be alter to make it consistent with the equation of continuity. Maxwell assumed that the definition of current density $\vec{J}$ is incomplete and hence $\vec{J}_d$ must be added to it. Then total current density which must be solenoidal, becomes $c = \vec{J} + \vec{J}_d$

Eqn (5) becomes.

$$\text{curl } \vec{H} = c = \vec{J} + \vec{J}_d \quad \text{——— (9)}$$

To identify $\vec{J}_d$:-

Take divergence of eqn (9),

$$\text{div curl } \vec{H} = \text{div } (\vec{J} + \vec{J}_d).$$

Since $\text{div curl } \vec{H} = 0$.

$$\therefore \text{div } (\vec{J} + \vec{J}_d) = 0$$

$$\text{or } \text{div } \vec{J} + \text{div } \vec{J}_d = 0$$

$$\Rightarrow \text{div } \vec{J} = -\text{div } \vec{J}_d \quad \text{——— (10)}$$

Since from continuity equation,

$$\text{div } \vec{J} = -\frac{\partial \rho}{\partial t}.$$

$$\therefore \text{div } \vec{J}_d = \frac{\partial \rho}{\partial t} \quad \text{——— (11)}$$

From Gauss's theorem in divergence differential form,

$$\vec{\nabla} \cdot \vec{D} = \rho.$$

$$\text{or } \text{div } \vec{D} = \rho \quad \text{——— (12)}$$

Using this in eqn (11) we get.

$$\text{div } \vec{J}_d = \frac{\partial}{\partial t} (\text{div } \vec{D})$$

$$= \text{div } \frac{\partial \vec{D}}{\partial t}$$

$$\Rightarrow \boxed{\vec{J}_d = \frac{\partial \vec{D}}{\partial t}} \quad \text{——— (13)}$$

$\therefore$ The modified form of Ampere's law is,

$$\text{Curl } \vec{H} = \vec{J} + \vec{J}_d = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{— (14)}$$

$$\text{or } \text{curl } \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \quad \text{——— (15)}$$

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$$\Rightarrow \text{curl } \vec{B} = \mu_0\left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

The term which Maxwell added to Ampere's law to include time varying fields is known as <u>displacement current</u> because it arises when electric displacement vector $\vec{D}$ changes with time. Maxwell assumed that ~~this~~ ~~term~~ displacement current is equally important as effective as conduction current $\vec{J}$ for producing magnetic field).

(Displacement current is a current only in the sense that it produces a magnetic field. It has none of the other properties of current since it is not linked with the moving motion of charges) For example displacement current has a finite value even in a perfect vacuum where there are no charges at all.

Displacement current in a good conductor is negligible as compared to the conduction current at any frequency less than optical frequencies ($\simeq 10^{15}$ Hz).

((With the help of displacement current Maxwell was able to derive his theory of electromagnetic waves))

Maxwell's equations

Four fundamental equations of electromagnetism known as Maxwell's equation (in differential form) as.

1. $\vec{\nabla} \cdot \vec{D} = \rho$ (differential form of Gauss law in electrostatics)

2. $\vec{\nabla} \cdot \vec{B} = 0$ (differential form of Gauss law in magnetostatics

3. $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (differential form of Faraday's law of electromagnetic induction)

4. $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

$\Rightarrow \vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right)$ (Maxwell's modification of Ampere's law)

Where $\vec{D}$ = electric displacement vector in coulomb/m^2

ρ = charge density in coul/m^3.

B = magnetic induction in weber/m^2.

$\bar{E}$ = Electric field intensity in volt/m.

$\bar{H}$ = magnetic field intensity in amp/m.

Derivation of Maxwell's equation:

1. Derivation of first equation $\text{div}\, \bar{D} = \vec{\nabla} \cdot \vec{D} = \rho$.

Let us consider a surface S bounding a volume V in a dielectric medium. In a dielectric medium total charge consists of free charge plus polarisation charge. If ρ & ρ_p are the charge densities of free charge and polarisation charge at a point in a small volume element dv, then Gauss law can be expressed as

$$\int_S \bar{E} \cdot d\vec{s} = \frac{1}{\epsilon_0} \int_V (\rho + \rho_p)\, dv$$

But polarisation charge density $\rho_p = -\text{div}\,\vec{P}$

$$\therefore \oint_S \vec{E}\cdot d\vec{s} = \frac{1}{\epsilon_0}\int_V (\rho - \text{div}\,\vec{P})\,dv$$

$$\Rightarrow \oint_S \epsilon_0 \vec{E}\cdot d\vec{s} = \int_V \rho\, dv - \int_V \text{div}\,\vec{P}\, dv$$

Using Gauss divergence theorem

$$\int_V \text{div}\,(\epsilon_0 \vec{E})\, dv = \int_V \rho\, dv - \int_V \text{div}\,\vec{P}\, dv$$

$$\Rightarrow \int_V \text{div}\,(\epsilon_0\vec{E} + \vec{P})\, dv = \int_V \rho\, dv \quad \text{——— (1)}$$

$$\Rightarrow \int_V \text{div}\,\vec{D}\, dv = \int_V \rho\, dv$$

Where $\vec{D} = \epsilon_0\vec{E} + \vec{P}$, electric displacement vector

$$\Rightarrow \int_V (\text{div}\,\vec{D} - \rho)\, dv = 0.$$

Since this equation is true for all volumes, therefore the integrand must vanish.

i,e $\text{div}\,\vec{D} - \rho = 0$

$\Rightarrow \text{div}\,\vec{D} = \rho$ i,e $\vec{\nabla}\cdot\vec{D} = \rho$.

2. Derivation of Second equation $\text{div}\,\vec{B} = \vec{\nabla}\cdot\vec{B} = 0$

Since isolated magnetic poles and magnetic currents due to them have no physical significance; therefore magnetic lines of force in general are either closed curves or go off to infinity. The number of magnetic lines of force entering any closed surface is exactly the same as leaving it. It means that the flux of magnetic induction

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$\vec{B}$ across any closed surface is always zero, i.e., $\int_S \vec{B}\cdot d\vec{s} = 0$

Using Gauss divergence theorem,

$$\int_V \operatorname{div}\vec{B}\, dv = 0$$

$$\Rightarrow \operatorname{div}\vec{B} = 0 \quad \text{or} \quad \vec{\nabla}\cdot\vec{B} = 0$$

3. <u>Derivation of third equation:</u>

$$\operatorname{curl}\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

According to Faraday's law of electromagnetic induction, induced emf in a closed loop is negative rate of change of magnetic flux,

$$e = -\frac{d\phi}{dt}$$

But magnetic flux, $\phi = \int_S \vec{B}\cdot d\vec{s}$ where S is any surface having loop as boundary.

$$e = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s}$$

$$= -\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{s} \longrightarrow (i)$$

(Since surface is fixed in space only B changes with time)

Since emf is also the work done in carrying a unit charge round the closed loop c. Thus if $\vec{E}$ is the electric field intensity at a small element $d\vec{l}$ of loop, we have

$$e = \oint_c \vec{E}\cdot d\vec{l} \longrightarrow (ii)$$

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Comparing the equations ② & ③,

$$\oint_c \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} \qquad (4)$$

Using Stoke's theorem,

$$\int_S \text{curl}\,\vec{E}\cdot d\vec{s} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

$$\Rightarrow \int_S \left(\text{curl}\,\vec{E} + \frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{s} = 0 \qquad (5)$$

Since surface is arbitrary, therefore eqn ⑤ holds only if the integrand vanishes,

$$\text{i.e}\quad \text{curl}\,\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\Rightarrow \text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{i.e}\quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

4. Derivation of fourth equation

$$\text{curl}\,\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}.$$

For this derivation refer
Displacement current: - - (-) ~~unit~~

Maxwell's Equation in Integral form.

1. Maxwell's first equation $\vec{\nabla}\cdot\vec{D} = \rho$.
Integrating over an arbitrary volume V,

$$\int_V \vec{\nabla}\cdot\vec{D}\,dV = \int_V \rho\,dV$$

Using Gauss's divergence theorem,

$$\int_S \vec{D}\cdot d\vec{S} = \int_V \rho\,dV$$

Where S is the surface which bounds

volume V. Since $\int_V \rho\, dv = q$, the net charge contained in volume V.

$$\therefore \int_S \vec{D}\cdot d\vec{s} = q \Rightarrow \int_S \vec{E}\cdot d\vec{s} = \frac{q}{\epsilon_0} \quad \text{—— (1)}$$

eqn (1) signifies that: The net outward flux of electric displacement vector through the surface enclosing a volume is equal to the net charge contained within that volume.

2. Maxwell's second equation is $\vec{\nabla}\cdot\vec{B} = 0$.

Integrating this over an arbitrary volume V,

$$\int_V \vec{\nabla}\cdot\vec{B} = 0.$$

Using Gauss's divergence theorem -

$$\int_S \vec{B}\cdot d\vec{s} = 0 \quad \text{—— (2)}$$

Where S is the surface which is bounded by volume V.

Eqn 2 signifies that: ∴ The net outward flux of magnetic induction $\vec{B}$ through any closed surface is equal to zero.

3. Maxwell's third equation is

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Integrating above equation over a surface S bounded by a curve C,

$$\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{s}$$

Using Stoke's theorem,

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int_S \vec{B}\cdot d\vec{s} \quad \text{—— (3)}$$

eqn (3) signifies that:
The electromotive force (e.m.f, $e = \int_C \vec{E} \cdot d\vec{l}$)

around a closed path is equal to negative rate of change of magnetic flux linked with the path (since magnetic flux $\phi = \int_S \vec{B} \cdot d\vec{s}$)

4. Maxwell's fourth equation is

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Taking surface integral over surface S bounded by curve C,

$$\int_S (\vec{\nabla} \times \vec{H}) \cdot d\vec{s} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{s}$$

Using stokes theorem,

$$\oint_C \vec{H} \cdot d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \cdot d\vec{s} \quad (4)$$

Eqn (4) signifies that:
The magnetomotive force (m.m.f $= \oint \vec{H} \cdot d\vec{l}$) around a closed path is equal to the conduction current plus displacement current through any surface bounded by the path.

Maxwell's equations in following cases:-

Maxwell's equations in differential form,

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla}\cdot\vec{B}=0$$
$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$
& $$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$

Case I: free space

volume charge density, $\rho_v=0$

Current density, $J=0$

$\therefore$ $$\vec{\nabla}\cdot\vec{D}=0$$
$$\vec{\nabla}\cdot\vec{B}=0$$
$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \quad \& \quad \vec{\nabla}\times\vec{H}=\frac{\partial\vec{D}}{\partial t}$$

With $\vec{D}=\epsilon_0\vec{E}$ and $\vec{B}=\mu_0\vec{H}$

Where ϵ_0 & μ_0 are absolute permittivity and permeability of free space respectively

Case II: Linear isotropic medium.

$$\vec{D}=\epsilon\vec{E} \quad \& \quad \vec{B}=\mu\vec{H}.$$

Where ϵ & μ are absolute permittivity and permeability of medium respectively

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon}$$

$$\vec{\nabla}\cdot\vec{B}=0$$
$$\vec{\nabla}\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t} \quad \& \quad \vec{\nabla}\times\vec{H}=\epsilon\frac{\partial\vec{E}}{\partial t}+\vec{J}$$

Case III: For harmonically varying fields:

When electromagnetic fields vary harmonically with time

i.e. $\vec{D}=\vec{D}_0 e^{i\omega t}$ and $\vec{B}=\vec{B}_0 e^{i\omega t}$.

Where $\vec{D}_0$ & $\vec{B}_0$ are peak values of $\vec{D}$ & $\vec{B}$ respectively.

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$$\frac{\partial \vec{D}}{\partial t} = \vec{D}_0\, i\omega e^{i\omega t} = i\omega \vec{D}.$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{B}_0\, i\omega e^{i\omega t} = i\omega \vec{B}$$

$\therefore$

$$\vec{\nabla}\cdot\vec{D} = \rho$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -i\omega\vec{B} \Rightarrow \vec{\nabla}\times\vec{E} + i\omega\vec{B} = 0$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + i\omega\vec{D} \Rightarrow \vec{\nabla}\times\vec{H} - i\omega\vec{D} = \vec{J}.$$

(6)

Plane Electromagnetic waves in Free space:

Maxwell's equations are

$$\left.\begin{aligned} \operatorname{div} \vec{D} &= \vec{\nabla}\cdot\vec{D} = \rho \\ \operatorname{div} \vec{B} &= \vec{\nabla}\cdot\vec{B} = 0 \\ \operatorname{curl} \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \text{and} \quad \operatorname{curl} \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \end{aligned}\right\} \quad \text{and} \quad \begin{aligned} \vec{B} &= \mu\vec{H} \\ \vec{D} &= \epsilon\vec{E} \\ \vec{J} &= \sigma\vec{E} \end{aligned} \qquad (1)$$

Free space is characterised by

$$\rho = 0, \ \sigma = 0, \ \mu = \mu_0 \text{ and } \epsilon = \epsilon_0 \qquad (2)$$

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Therefore Maxwell's equations reduce to

$$\left.\begin{aligned} \operatorname{div}\vec{E} &= 0 \quad \text{—— (a)} \\ \operatorname{div}\vec{H} &= 0 \quad \text{—— (b)} \\ \operatorname{curl}\vec{E} &= -\mu_0 \frac{\partial \vec{H}}{\partial t} \quad \text{—— (c)} \\ \text{and } \operatorname{curl}\vec{H} &= \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad \text{—— (d)} \end{aligned}\right\} \text{—— (3)}$$

Taking curl of eqn 3(c), we get

$$\operatorname{curl}\operatorname{curl}\vec{E} = -\mu_0 \frac{\partial}{\partial t}\operatorname{curl}\vec{H}$$

$$\Rightarrow \operatorname{curl}\operatorname{curl}\vec{E} = -\mu_0 \frac{\partial}{\partial t}\left(\epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \qquad \text{(from eqn 3(d))}$$

$$\Rightarrow \operatorname{curl}\operatorname{curl}\vec{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{—— (4)}$$

$$\Rightarrow \operatorname{grad}\operatorname{div}\vec{E} - \nabla^2\vec{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow -\nabla^2\vec{E} = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \qquad (\because \operatorname{div}\vec{E} = 0 \text{ from eqn 3(a)})$$

$$\Rightarrow \nabla^2\vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad \text{—— (5)}$$

Now taking curl of eqn 3(d), we get

$$\operatorname{curl}\operatorname{curl}\vec{H} = \epsilon_0 \frac{\partial}{\partial t}\operatorname{curl}\vec{E}$$

$$\Rightarrow \operatorname{curl}\operatorname{curl}\vec{H} = \epsilon_0 \frac{\partial}{\partial t}\left(-\mu_0 \frac{\partial \vec{H}}{\partial t}\right) \qquad \text{(from eqn 3(c))}$$

$$\Rightarrow \operatorname{grad}\operatorname{div}\vec{H} - \nabla^2\vec{H} = -\epsilon_0 \mu_0 \frac{\partial^2 \vec{H}}{\partial t^2} \quad \text{—— (6)}$$

$$\Rightarrow -\nabla^2\vec{H} = -\epsilon_0 \mu_0 \frac{\partial^2 \vec{H}}{\partial t^2} \qquad (\because \operatorname{div}\vec{H} = 0 \text{ from eqn 3(b)})$$

$$\Rightarrow \nabla^2\vec{H} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \quad \text{—— (7)}$$

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involving Equations (5) & (7) represent wave equations governing electromagnetic fields $\vec{E}$ and $\vec{H}$ in free space. Equations (5) & (7) are vector equations of identical form which means that each of the six components of $\vec{E}$ & $\vec{H}$ separately satisfies the same scalar wave equation of the form.

In general

$$\nabla^2 u - \mu_0 \epsilon_0 \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{——— (8)}$$

Where u is a scalar and which can stand for represent one of the components of $\vec{E}$ and $\vec{H}$. It resembles with the general wave equation

$$\nabla^2 u = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} \quad \text{——— (9)}$$

Where v is the velocity of wave.

Comparing eqn (8) & (9) we see that the field vectors $\vec{E}$ and $\vec{H}$ are propagated in free space as waves at a speed equal to

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Since $\mu_0 = 4\pi \times 10^{-7}$ weber/Amp-m

$\epsilon_0 = 8.542 \times 10^{-12}$ farad/m

$$= \sqrt{\frac{4\pi}{\mu_0 4\pi\epsilon_0}}$$

since $\frac{1}{4\pi\epsilon_0} = 9 \times 10^9$ m/farad.

$$\therefore v = \sqrt{\frac{\cancel{4\pi}}{\cancel{4\pi} \times 10^{-7}} \times 9 \times 10^9} = 3 \times 10^8 \text{ m/s}$$

$= c$, speed of light

$\therefore$ We write c the speed of light in place of $\frac{1}{\sqrt{\mu_0 \epsilon_0}}$. So eqns (5), (7) can be written as

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad \text{——— (10)}$$

$$\nabla^2 \vec{H} - \frac{1}{c^2}\frac{\partial^2 \vec{H}}{\partial t^2} = 0 \quad \text{——— (11)}$$

and $$\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \quad \text{——— (12)}$$

Now let us find the solution of above equations for plane electromagnetic waves. A plane wave is defined as a wave whose amplitude is the same at any point in a plane perpendicular to a specified direction.

The plane wave solutions of above equations can be written as.

$$\vec{E}(\vec{r},t) = \vec{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad \text{——— (13)}$$

$$\vec{H}(\vec{r},t) = \vec{H}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad \text{——— (14)}$$

$$u(\vec{r},t) = u_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} \quad \text{——— (15)}$$

where $\vec{E}_0$, $\vec{H}_0$ and u_0 are complex amplitudes which are constant in space and time while $\vec{k}$ is a wave propagation vector denoted as.

$$\vec{k} = k\hat{n} = \frac{2\pi}{\lambda}\hat{n} = \frac{2\pi\nu}{c}\hat{n} = \frac{\omega}{c}\hat{n} \quad \text{(16)}$$

Here $\hat{n}$ is a unit vector in the direction of wave propagation. Now in order to apply the condition $\vec{\nabla}\cdot\vec{E} = 0$ and $\vec{\nabla}\cdot\vec{H} = 0$, Let us first find $\vec{\nabla}\cdot\vec{E}$ and $\vec{\nabla}\cdot\vec{H}$

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$$\vec{\nabla}\cdot\vec{E} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\vec{E}_0\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(\hat{i}E_{0x}+\hat{j}E_{0y}+\hat{k}E_{0z})\, e^{i(k_x x+k_y y+k_z z)-i\omega t}\right]$$

$$= (E_{0x}ik_x + E_{0y}ik_y + E_{0z}ik_z)\, e^{i\vec{k}\cdot\vec{r}-i\omega t}$$

$$= i(k_x E_{0x} + k_y E_{0y} + k_z E_{0z})\, e^{i\vec{k}\cdot\vec{r}-i\omega t}$$

$$= i(k_x\hat{i}+k_y\hat{j}+k_z\hat{k})\cdot(\hat{i}E_{0x}+\hat{j}E_{0y}+\hat{k}E_{0z})\, e^{i\vec{k}\cdot\vec{r}-i\omega t}$$

$$= i\vec{k}\cdot\vec{E}_0\, e^{i\vec{k}\cdot\vec{r}-i\omega t} = i\vec{k}\cdot\vec{E}$$

Similarly $\vec{\nabla}\cdot\vec{H} = i\vec{k}\cdot\vec{H}$

Thus the requirements $\vec{\nabla}\cdot\vec{E}=0$ & $\vec{\nabla}\cdot\vec{H}=0$ demand that

$$\vec{k}\cdot\vec{E}=0$$
$$\text{and } \vec{k}\cdot\vec{H}=0 \qquad (17)$$

This means that electromagnetic field vectors $\vec{E}$ and $\vec{H}$ are both perpendicular to the direction of propagation vector $\vec{k}$. This implies that electromagnetic waves are transverse in nature.

Now $\text{curl}\,\vec{E} = -\mu_0\frac{\partial\vec{H}}{\partial t}$ and $\text{curl}\,\vec{H} = \epsilon_0\frac{\partial\vec{E}}{\partial t}$

using eqns (13) & (14) above equations gives:

$$i\vec{k}\times\vec{E} = -\mu_0(-i\omega\vec{H}) \text{ or } \vec{k}\times\vec{E} = \mu_0\omega\vec{H} \qquad (18)$$

$$\text{and } i\vec{k}\times\vec{H} = \epsilon_0(-i\omega\vec{E}) \text{ or } \vec{k}\times\vec{H} = -\epsilon_0\omega\vec{E} \qquad (19)$$

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From eqn (18) it is clear that field vector $\vec{H}$ is perpendicular to both $\vec{k}$ and E and from eqn (19), it is clear that field vector $\vec{E}$ is perpendicular to both $\vec{k}$ and $\vec{H}$. This simply means that field vectors $\vec{E}$ and $\vec{H}$ are mutually perpendicular and also they are perpendicular to the direction of propagation of wave.)

This in turn implies that in a plane electromagnetic wave, vectors $(\vec{E}, \vec{H}, \vec{k})$ form a set of orthogonal vectors which form a right-handed coordinate system in the order.

From equation (18) -

$$\vec{H} = \frac{1}{\mu_0 \omega}(\vec{k}\times\vec{E}) = \frac{k}{\mu_0 \omega}(\hat{n}\times\vec{E})$$

$$= \frac{1}{\mu_0 c}(\hat{n}\times\vec{E}) \quad \left(\because \vec{k} = k\hat{n} \;\&\; \frac{\omega}{k} = c\right)$$

$$= \frac{1}{\mu_0 c}\vec{E} \text{ ——— (20) (in terms of moduli)}$$

where $\frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$.

Now wave impedance;

$$\vec{Z}_0 = \frac{\text{~~ratio of~~ magnitude of } \vec{E}}{\text{magnitude of } \vec{H}}$$

$$= \left|\frac{\vec{E}}{\vec{H}}\right| = \left|\frac{\vec{E}_0}{\vec{H}_0}\right| = \mu_0 c = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

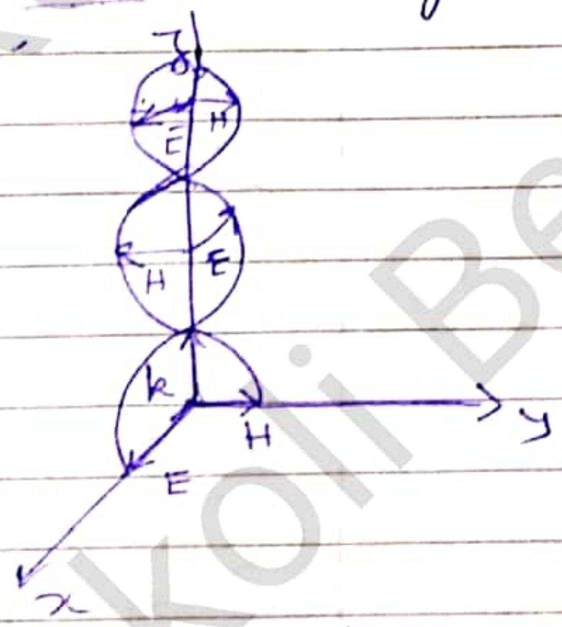
since $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

$$= \sqrt{\frac{4\pi\times10^{-7}}{8.85\times10^{-12}}}$$

$$= 376.6 \text{ ohms. ——— (21)}$$

(units of $Z_0 = \frac{\text{volt/m}}{\text{amp-turn/m}} = \frac{\text{volt}}{\text{amp}} = \text{ohm}$)

Since the unit of $\frac{E}{H}$ are the same as those of Impedance ~~of free space~~, the value of $\vec{Z_0}$ is called as wave impedance of free space. Also since the ratio $\vec{Z_0} = \left|\frac{\vec{E}}{\vec{H}}\right|$ is real and positive : This implies that field vectors $\vec{E}$ and $\vec{H}$ are in the same phase i.e., they have the same relative magnitude at all points at all times.



Ratio of Electrostatic and magnetic energy densities is given by

$$\frac{u_e}{u_m} = \frac{\frac{1}{2}\epsilon_0 E^2}{\frac{1}{2}\mu_0 H^2} = \frac{\epsilon_0}{\mu_0}\frac{E^2}{H^2} = \frac{\epsilon_0}{\mu_0}\cdot\frac{\mu_0}{\epsilon_0} = 1. \qquad (23)$$

Since $\frac{E}{H} = \sqrt{\frac{\mu_0}{\epsilon_0}}$

i.e. the electro~~magnetic~~static energy density is equal to magnetostatic energy density.

Total electromagnetic energy density

$\therefore\ u = u_e + u_m = 2u_e = 2\cdot\frac{1}{2}\epsilon_0 E^2 = \epsilon_0 E^2$

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i.e. Time average of energy density

$$\langle u \rangle = \langle \epsilon_0 E^2 \rangle = \epsilon_0 \langle \{\vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}\}^2_{real} \rangle$$

$$= \epsilon_0 E_0^2 \langle \cos^2(\omega t - \vec{k}\cdot\vec{r}) \rangle$$

$$= \frac{1}{2}\epsilon_0 E_0^2 = \epsilon_0 E_{rms}^2 \qquad (24)$$

Summarising for electromagnetic waves in free space

1) In free space, the electromagnetic waves travel with the speed of light.

2) The electromagnetic field vectors $\vec{E}$ & $\vec{H}$ are mutually perpendicular and they are also perpendicular to the direction of

propagation of electromagnetic waves. Thereby indicating the electromagnetic waves are transverse in nature.

3) The field vectors $\vec{E}$ and $\vec{H}$ are in same phase.

4) The direction of flow of electromagnetic energy is along the direction of wave propagation and the energy flow per unit area per second is given by

$$\langle \vec{S} \rangle = \frac{E_{rms}^2}{Z_0}\hat{n} = \langle u \rangle c\,\hat{n}.$$

5) The electrostatic energy density is equal to the magnetic energy density and the energy density associated with the electromagnetic wave in free space propagates with the speed of light.

$\langle u \rangle = \epsilon_0 E_{rms}^2$