B.A.,Sem-II,Mathematics(Algebra)
Field

The set $R = \{0, 1, 2, 3, 4, 5\}$ is a commutative ring with respect to $'+_6'$ and $'\times_6'$ as the two ring compositions. In the above problem, when we form the composition table for '$\times_6$', we get $2 \times_6 3 = 0$ while neither 2 nor 3 is equal to the zero element of the ring.Thus in a ring it is possible that the product of two non-zero elements is equal to the zero element.
Ring with zero divisor: If in a ring, it is possible that the product of two non-zero elements of the ring is equal to the zero element,then this ring is called ring with zero divisor.

Ring without zero divisor:A ring is without zero divisor if it is not a ring with zero divisor.

Note: The ring of integers I is a ring without zero divisors because the product of two non-zero integers is not equal to zero integer.

Now,we turn to a new structure called **Field**,which is needful for the study of a Vector space in next unit.

Definition 1 *A commutative ring R having identity is called a **field** if each non-zero element possesses multiplicative inverse.*
OR
If a field is denoted by F and all the axioms of the field are listed, then the field can be defined as follows:
Let F be a set with two binary operations $'+'$ and $'.'$. Then F is called a field if

1. *Addition is associative. For all $a, b, c \in F$*
   $a + (b + c)$=$(a + b) + c$ *and* $a.(b.c)$=$(a.b).c$.

2. *Addition is commutative. For all $a, b \in F$*
   $a + b = b + a$.

3. *Additive identity.The set F contains an element denoted by 0 such that*
   $a + 0 = 0 + a = a$ *for all* $a \in F$

4. *Additive inverse For each $a \in F$,the equations*
   $a + x = 0$ *or* $x + a = 0$
   have a solution in F,called the additive inverse of a and denoted by $-a$.

5. *Multiplication is associativei.e. $a.(b.c) = (a.b).c$ for all $a, b, c \in F$*

6. *multiplication is commutative For all $a, b \in F$,we have*
   $a.b = b.a$

7. *if it contains an element 1 such that for all $a \in F$,*
   $a.1 = 1.a = a$. *Here 1 is called multiplicative identity element.*

8. *multiplicative Inverse To each non-zero element a in F, there exist an element b in F such that $ab = 1$, where 1 is the Unity or identity element of F.*

9. *Distributive laws. For all $a, b, c \in F$,*
   $a.(b + c) = a.b + a.c$ *and* $(a + b).c = a.c + b.c$.

Note: *A field has no zero divisors.Therefore in a field, the product of two non-zero elements will again be a non-zero element of the field.*
From the above properties of a field, It is observed that if F is a field then the set F form an abelian group with respect to addition composition and the set of all non-zero elements of the field also form an abelian group under the multiplication composition.
There are some examples of the field.The students are advised to verify all the axioms of the field for each example.

1. *The set Q of rational numbers with respect to ordinary addition and multiplication.*

2. *The set R of real numbers with respect to ordinary addition and multiplication.*

3. *The set of numbers of the form $a + b\sqrt{2}$, where a and b are rational numbers with respect to ordinary addition and multiplication.*

4. *Let C be the set of the ordered pairs (a, b) of real numbers. The addition and multiplication in C are defined as*
$(a, b) + (c, d) = (a + c, b + d)$
$(a, b)(c, d) = (ac - bd, bc + ad)$.

5. *The set $Z_p = \{0, 1, 2, 3, ...p - 1\}$ with respect to addition modulo $+_p$ and multiplication modulo $\times_p$, where p is a prime number.*