Unit V   Maxwell Equations

(i) $\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0$ (Gauss law for electrostatics)

(ii) $\vec{\nabla} \cdot \vec{B} = 0$ (Net magnetic flux for closed surface is always zero)

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday's law for Electromagnetic Induction)

(iv) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

(Ampere's law with maxwell modification)

where
$\rho \rightarrow$ volume charge density
$\varepsilon_0 \rightarrow$ Permitivity of free space.
$\vec{J} \rightarrow$ Current density
$\mu_0 \rightarrow$ Permeability of free space

Displacement vector $\vec{D} = \varepsilon_0 \vec{E}$

$\vec{B} = \mu_0 \vec{H}$

or

Magnetic field intensity $\vec{H} = \frac{B}{\mu_0}$

Maxwell Equations in terms of Displacement Vector $\vec{D}$ and $\vec{H}$.

(i) $\vec{\nabla}\cdot\vec{D} = \rho$

(ii) $\vec{\nabla}\cdot\vec{H} = 0$

(iii) $\vec{\nabla}\times\vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

(iv) $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \mu_0 \frac{\partial \vec{D}}{\partial t}$

(i) $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$

Proof :- According to Gauss law of Electrostatic

$$\phi_E = \oint \vec{E}\cdot\vec{ds} = \frac{q}{\varepsilon_0} \quad \text{—(1)}$$

By using Gauss divergence theorem we can write

$$\oint_S \vec{E}\cdot\vec{ds} = \int_V (\vec{\nabla}\cdot\vec{E})\, dV$$

therefore (1) ⇒

$$\int (\vec{\nabla}\cdot\vec{E})\, dV = \frac{q}{\varepsilon_0} \quad \text{—(2)}$$

Now from defination of volume charge density $\rho = \frac{dq}{dV}$ or $q = \int \rho\, dV$

So (2) $\Rightarrow$

$$\int_V (\vec{\nabla} \cdot \vec{E})\, dV = \frac{1}{\varepsilon_0} \int \rho \, dV$$

By comparing both sides we ~~can~~ have

$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}$$

Derivations :-

Maxwell's Second Equation :-

We know that magnetic flux is defined as

$$\phi_B = \oint_S \vec{B} \cdot d\vec{S} \quad \text{--- (1)}$$

for magnetic ~~flu~~ field, magnetic field lines always make closed loop i.e there is no source or sink. therefore net magnetic flux is always zero. So we can write

(1) $\Rightarrow$

$$\oint \vec{B} \cdot d\vec{S} = 0 \quad \text{--- (2)}$$

Now By using Gauss divergence theorem

$$\oint \vec{B} \cdot d\vec{S} = \int (\vec{\nabla} \cdot \vec{B})\, dV$$

therefore (2) $\Rightarrow$

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0}$$

Maxwell Second eqn.

Maxwell Third Equation :-

$$\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$$ (faraday's law of electromagnetic Induction)

Proof :

According to faraday ~~first~~ law of electromagnetic Induction, induced emf is equal to rate of change of magnetic flux i.e

$$e = -\frac{d\phi}{dt}$$

e.m.f is defined as $\int \vec{E} \cdot \vec{dl}$

$$\int \vec{E} \cdot \vec{dl} = -\frac{d\phi}{dt}$$

$$\int \vec{E} \cdot \vec{dl} = -\frac{d}{dt} \int \vec{B} \cdot \vec{ds}$$

By Stokes curl theorem

$$\int (\vec{\nabla} \times \vec{E})\, ds = -\frac{d}{dt} \int \vec{B}\, \vec{ds}$$

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}}$$

(By comparing both sides)

H.P.

Maxwell's IVth Equation:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{d\vec{E}}{dt}$$

(Ampere's law with Maxwell modification)

Proof:-

According to Ampere's circuital law

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I \quad \text{—— (1)}$$

(Line integral of magnetic field is μ_0 times the total current I).

By stokes curl theorem

$$\oint \vec{B} \cdot \vec{dl} = \int (\vec{\nabla} \times \vec{B})\, ds \quad \text{—— (2)}$$

and

$$I = \int \vec{J}\, ds \quad \text{—— (3)}$$

where $\vec{J} \rightarrow$ current density

Now from (1) we have

$$\int (\vec{\nabla} \times \vec{B})\, \vec{ds} = \mu_0 \int \vec{J}\, ds$$

$$\therefore \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad \text{—— (4)}$$

(Integral form of Ampere's law)

By taking divergence of both sides of eqⁿ ④ we get

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0 \vec{\nabla}\cdot\vec{J} \quad \text{—⑤}$$

The left hand side of the equation being a divergence of a curl vanishes, So we have

$$\vec{\nabla}\cdot\vec{J} = 0$$

However we know from the continuity equation

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Clearly in the presence of time dependent charge densities, the divergence of current density is non-vanishing, However Ampere's law demands that the divergence of $\vec{J}$ must vanish. i.e there is a contradiction. pointed out by maxwell.

The problem was solved by maxwell by introducing ~~the~~ term displacement 'current'.

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$ (Maxwell Ist eqⁿ or differential form of Gauss law)

Continutity Equation.

$$0 = \vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t}$$

$$= \vec{\nabla} \cdot \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

Notice that we have a quantity $\left(J + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$ whose divergence is zero.

therefore

$$\vec{J}_{tot} = \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

therefore Ampere's law

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

or

$$\boxed{\begin{aligned} \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \\ \nabla \times H &= \vec{J} + \frac{\partial \vec{D}}{\partial t} \end{aligned}}$$

$\vec{D} \rightarrow$ Displacement vector.

$\frac{\partial D}{\partial t} \rightarrow$ Term due to displacement current

→ Displacement current occurs due to time varying field.

Physical Significance of Maxwell Equations :

(1) $\vec{\nabla}.\vec{E} = \rho/\varepsilon_0$

or $\vec{\nabla}.\vec{D} = \rho$

→ It is time independant equation.

→ Since $\vec{\nabla}.\vec{E}$ is scalar, therefore charge density ρ is also scalar quantity.

→ It is statement of Gauss law of electrostatics.

(II) $\vec{\nabla}.\vec{B} = 0$

→ It is time independant equation.

→ Accordng to this equation isolated magnetic poles do not exist.

→ Since $\oint \vec{B}.\vec{ds} = 0$, i.e. number of lines of magnetic force leaving and entry a given volume are equal.

→ It is statement of Gauss law in Magnetism.

(III) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

→ It is time dependant equation

→ It implies that time variation of magnetic field generates electric field.

→ It is statement of Faraday's law of electromagnetic induction and negative sign justifies Lenz's law

(IV) $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

or $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

→ It is time dependant equation

→ It shows that magnetic field can be genrated by current density vector and ~~e4r~~ time variation of $\vec{D}$.

→ It relates magnetic field vector ($\vec{B}$) with electric displacement vector ($\vec{D}$) and current density vector ($\vec{J}$).

→ It is statement of Ampere's law with Maxwell modification.

Scanned with CamScanner